MATH 5343
Test II Name___________________________

1. Consider the method ($U_i^k = U(x_i, t_k)$)

$$\frac{U_i^{k+1}-U_i^k}{dt} = (1-\alpha)\frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{dx^2} + \alpha\frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

   a. For what value of α is the truncation error of highest order, when this is used to approximate $u_t = u_{xx}$? (no need to justify answer)

   b. Use the Fourier method to analyze stability, that is, plug in $U_i^k = a(t_k)e^{Imx_i}$. For what values of α is this method unconditionally stable? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m * dx/2)$)

   c. For what values of α is the method implicit?

2. Write a second order, centered difference approximation to the PDE $u_{xx} + u_{yy} + yu_x + 2u = sin(x)$ and calculate its truncation error.

3. Consider the boundary value problem $u^{iv}(x) = f(x)$.

   a. Develop a second order finite difference approximation by starting from the approximation:

   $$u^{iv}(x_i) \approx \frac{u''(x_{i+1})-2u''(x_i)+u''(x_{i-1})}{dx^2}$$

   and replace each second derivative by a centered difference approximation.

   b. Write out the Successive Overrelaxation iteration, for solving this system of equations.

   c. Which method will solve this system faster, a band solver or SOR?

4. Consider the damped wave equation
   $u_{tt} + au_t = c^2 u_{xx} + f(x,t)$
   with initial conditions
   $u(x,0) = h(x)$
   $u_t(x,0) = q(x)$

   An explicit, centered finite difference approximation will require starting values at the first two time levels, that is, we need to know $u(x,0)$ and $u(x,dt)$. Obviously we can take $u(x,0) = h(x)$; suggest a reasonable formula to use for $u(x,dt)$, which will preserve the second order accuracy of the approximation.

5. True or False:

   a. It is impossible to design explicit methods which are unconditionally stable and consistent with the heat equation ($u_t = Du_{xx}$).

   b. The stability criteria for the usual explicit finite difference approximation to the wave equation ($u_{tt} = c^2 u_{xx}$) is less severe than the stability criterion for the usual explicit finite difference approximation to the heat equation, hence explicit methods are more popular for approximating the wave equation than the heat equation.

   c. If the exact solution of a PDE at a point (x, t) depends on the initial conditions over a certain interval, while, as dt and $dx \to 0$, the finite difference approximations at that point all depend on the initial conditions over a proper subset of this interval, the finite difference method cannot be stable (assume it is consistent).

   d. Same as (c) but replace "proper subset" with "proper superset".

   e. For the shifted inverse power method, choosing the shift parameter closer to an eigenvalue will make the method converge more rapidly to this eigenvalue, generally.

   f. Nonlinear successive overrelaxation, with $\omega = 1$, is equivalent to Newton's method with the Jacobian matrix approximated by its diagonal part.

   g. For the transport (convection only) equation, the boundary conditions should be specified "downwind".

   h. The explicit upwind approximation to the transport equation is unconditionally stable.

   i. If the initial conditions are discontinuous, the solution to the diffusion/convection equation $u_t = Du_{xx} - vu_x$ is continuous everywhere for $t > 0$, assuming $D > 0$; but if $D = 0$ (ie, there is now no diffusion, only convection) the solution may still be discontinuous for $t > 0$.

   j. If the shifted inverse power method is used to find an eigenvalue of a matrix, with fixed shift, the LU decomposition found on the first iteration can be used on subsequent iterations to decrease the computer time for these iterations.