MATH 5343

Test II Name___________________________

1. Consider the method ($U_i^k = U(x_i, t_k)$)

$$\frac{U_i^{k+1} - U_i^{k-1}}{2dt} = \frac{U_{i+1}^k - (U_i^{k+1} + U_i^{k-1}) + U_{i-1}^k}{dx^2}$$

   a. Calculate the truncation error, when this is used to approximate $u_t = u_{xx}$. What can you say about the consistency of this method?

   b. Use the Fourier method to analyze stability, that is, plug in $U_i^k = a(t_k)e^{Imx_i}$. (Hint: $e^{Imdx} + e^{-Imdx} = 2cos(m * dx)$)

2. Suppose the centered finite difference method:

$$\frac{U_i^{k+1}-2U_i^k+U_i^{k-1}}{dt^2} = c^2\frac{U_{i+1}^k-2U_i^k+U_{i-1}^k}{dx^2}$$

is used to approximate $u_{tt} = c^2 u_{xx}$. This models motion of a wave with speed c, so it is known that the solution at a point (x, t) depends on the initial conditions at points within a distance ct of x, that is, on the initial contitions in the interval $(x - ct, x + ct)$. **Using this information only**, what can you conclude about the relationship between dt and dx required for stability?

3. Consider the problem $u_{xxxx} = u^3 + 1$ in $0 < x < 1$, with boundary conditions $u(0) = u_x(0) = u_x(1) = u(1) = 0$.

   a. Develop a second order finite difference approximation for u_{xxxx} by starting from the approximation:

   $$u_{xxxx}(x_i) \approx \frac{u_{xx}(x_{i+1})-2u_{xx}(x_i)+u_{xx}(x_{i-1})}{dx^2}$$

   and replace each second derivative by a centered difference approximation.

   b. Below is a Fortran90 program designed to solve the boundary value problem stated above, with $u_{xxxx}(x_i)$ approximated by the finite difference formula from part (a). Here u(i) is the solution at x(i) = i*h, and we approximate the boundary conditions using $u(0) = 0, u(1) - u(-1) = 0, u(n) = 0, u(n + 1) - u(n - 1) = 0$.
   The nonlinear relaxation method is used (Newton's method, with the Jacobian replaced by its diagonal, and with a parameter w) in this program. Finish the incomplete statement (Fortran syntax does not have to be correct!)

```
parameter (n=50)
double precision u(-1:n+1)
niter = 10000
w = 1.9
h = 1.d0/n
u(-1:n+1) = 0
do iter=1,niter
   do i=1,n-1
      u(i) =




   end do
   u(-1) = u(1)
   u(n+1) = u(n-1)
end do
print *, u
stop
end
```

4. a. Find the eigenvalues of $u'' = \lambda u$ with $u'(0) = u'(\pi) = 0$. (Hint: the eigenfunctions are $u_k(x) = cos(kx)$.

b. Approximate this with a centered finite difference problem, and find the eigenvalues of this discrete problem. (Hints: the eigenvectors are $U_k(x_i) = cos(kx_i)$, where $x_i = ih, h = \pi/N$; also, $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$ and $1 - cos(\theta) = 2sin^2(\theta/2)$.)

c. Show that for fixed k, as $h \rightarrow 0$, the k^{th} eigenvalue of the finite difference problem (b) converges to the k^{th} eigenvalue of the PDE problem (a).