MATH 5343

Test II Name___________________________

1. Consider the method ($U_i^k = U(x_i, t_k)$)

$$\frac{U_i^{k+1} - U_i^k}{dt} = \frac{U_i^k - U_{i-1}^k}{dx}$$

a. Calculate the truncation error, when this is used to approximate $u_t = u_x$. What can you say about the consistency of this method?

b. Use the Fourier method to analyze stability, that is, plug in $U_i^k = a_k e^{Imx_i}, I = \sqrt{-1}$.

2. Explain why the fact that the exact solution at $(x, t), t > 0$ of the heat equation $u_t = Du_{xx}, u(x, 0) = h(x)$ depends on the initial conditions at all points x, shows that the usual explicit finite difference method cannot possibly be stable if dt and dx go to 0 with any constant ratio $dt/dx = r$.

3. Analyze the stability of the following approximation to $u_{tt} = c^2 u_{xx}$:

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = c^2 \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

Hint: $e^{\theta} - 2 + e^{-\theta} = -4sin^2(\theta/2)$

4. a. Calculate the truncation error for the following approximation to $u_{xxxx} = 1$, where $U_i = U(x_i)$.

$$\frac{U_{i+2} - 4U_{i+1} + 6U_i - 4U_{i-1} + U_{i-2}}{dx^4} = 1$$

b. Assume the boundary conditions for (a) are $u(0) = u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0$. Then (a) is a system of linear equations of the form $Au = f$. Given that the eigenvectors of the matrix A are $W_i = sin(mx_i)$, for integer m, find the eigenvalues. Hint: $sin(m(x+dx))+sin(m(x-dx)) = 2sin(mx)cos(m\ dx)$ and $cos(2\theta) = 2cos^2(\theta) - 1$.

c. Write out the Gauss-Seidel iteration to solve the finite difference equations of (a).

d. Is the Gauss-Seidel iteration of part (c) guaranteed to converge? Justify your answer.

5. True or False:

   a. Choosing the shift parameter closer to an eigenvalue will generally make the shifted inverse power method converge more rapidly to this eigenvalue.

   b. If SOR is applied to solve the equations of Problem 4a, it is guaranteed to converge for $0 < \omega < 2$.

   c. It is possible to apply the shifted inverse power method to the generalized eigenvalue problem $Az = \lambda Bz$, where A and B are banded matrices, without having to deal with full matrices.

   d. Successive overrelaxation applied to $Ax = b$, with $\omega = 1$, is guaranteed to converge if the matrix A is diagonal dominant or positive definite or negative definite.

   e. For the transport equation $u_t = -\nabla \bullet (u\mathbf{v})$, the boundary conditions should be specified on the part of the boundary where $\mathbf{v} \bullet \mathbf{n}$ is positive.

   f. The explicit upwind approximation to the transport equation is unconditionally stable.

   g. For the diffusion equation $u_t = Du_{xx}$, the speed of diffusion is infinite, while for the wave equation $u_{tt} = c^2 u_{xx}$ the velocity is finite.

   h. If the shifted inverse power method is used to find an eigenvalue of a matrix, with fixed shift, the LU decomposition found on the first iteration can be used on subsequent iterations to decrease the computer time for these iterations.

   i. If an implicit finite difference method is used to solve $U_t = U_{xx} + U_{yy} + U_{zz}$, you will need to solve a large linear system each time step which is banded, but sparse even inside the band.

   j. Nonlinear overrelaxation generally takes more work per iteration than Newton's method, but fewer iterations to converge.