

1. Under what conditions is the following method (to approximate  $u_{tt} = u_{xx}$ ) stable, where  $U_i^k \equiv U(x_i, t_k)$  (justify answer)? (Hint:  $e^{Imdx} - 2 + e^{-Imdx} = -4\sin^2(m * dx/2)$ )

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k-1} - 2U_i^{k-1} + U_{i-1}^{k-1}}{dx^2}$$

2. a. When a Galerkin finite element method is used to approximate the solution of  $u'' + f(x) = 0$ ,  $u(0) = u(1) = 0$ , and the approximate solution is expanded as a linear combination of piecewise linear "chapeau" trial functions  $\phi_i(x) : U(x) = \sum_{i=1}^{N-1} a_i \phi_i(x)$  there results a linear system  $\mathbf{A}\mathbf{a} = \mathbf{b}$  for the unknowns  $\mathbf{a} = (a_1, \dots, a_{N-1})$ . Give formulas (involving integrals, and only first derivatives) for the elements  $A_{ki}, b_k$  of the matrix and right hand side vector.

- b. If the integrals in  $A_{ki}$  are calculated exactly, and a mid-point rule is used to approximate the integral in  $b_k$ , taking into account that  $a_i = U(x_i)$ , the resulting linear equations simplify (assuming uniformly spaced  $x_i$ ) to:

$$\frac{-U(x_i+h) + 2U(x_i) - U(x_i-h)}{h} = \frac{h}{2} [f(x_i + h/2) + f(x_i - h/2)]$$

Find the truncation error for this "finite difference" scheme.

- c. Explain why the collocation method cannot be used here in place of the Galerkin method.
3. The usual centered, second order, finite difference approximations are used to solve  $u_{xx} + u_{yy} + u_{zz} = f(x, y, z)$ , with  $u = 0$  on the boundary of the square  $0 < x < 1, 0 < y < 1, 0 < z < 1$ . If  $M$  gridlines are used in each direction, ie,  $x_i = i/M, i = 0, \dots, M$  and similarly for  $y_j, z_k$ , a linear system results with  $(M - 1)^3$  unknowns and  $(M - 1)^3$  equations (one equation centered on each unknown). If the unknowns are numbered by planes, that is, all unknowns on the plane  $z_1$  are numbered first, then those on plane  $z_2$ , etc, then:
- a. Approximately how many multiplications are done if this linear system is solved by a Gauss elimination routine, which does not take any advantage of zero elements?
  - b. Approximately what is the half-bandwidth of the linear system? (Half-bandwidth =  $\max |i - j|$  such that  $A_{ij} \neq 0$ .)
  - c. Approximately how many multiplications are done if it is solved by a band solver, with partial pivoting?
  - d. Approximately how many if a band solver is used with no pivoting?
  - e. Approximately how many "divisions" are done if the SOR iterative method is used with optimal  $\omega$ , assuming the number of iterations is then about  $cM$ .

4. Consider the eigenvalue problem  $u_{xx} + u_{yy} = \lambda u$  with  $u = 0$  on the boundary of a 2D region  $R$ . An approximate eigenfunction of the form  $U(x, y) = \sum_{i=1}^N a_i \phi_i(x, y)$  is sought, where the  $\phi_i$  are linearly independent and each is 0 on the boundary.
- If a Galerkin finite element method is used, a generalized eigenvalue problem  $A\mathbf{a} = \lambda B\mathbf{a}$  results, where  $\mathbf{a} = (a_1, \dots, a_N)$ . Give formulas (involving integrals, and only first derivatives) for the elements  $A_{ki}, B_{ki}$  of matrices  $A$  and  $B$ .
  - Show that the matrix  $B$  in problem 4a is positive definite. (Recall that one definition of positive definite is that  $B$  is symmetric and  $z^T B z > 0$  for any nonzero vector  $z$ .) Thus,  $B$  is nonsingular.
  - Write the inverse power method iteration for finding the smallest eigenvalue of  $Az = \lambda Bz$  in a form where no inverses are calculated.
  - If a collocation finite element method is used, with collocation points  $(x_k, y_k)$ , a generalized eigenvalue problem  $Aa = \lambda Ba$  again results; this time what are the elements of  $A_{ki}, B_{ki}$  of  $A$  and  $B$ ? Is  $B$  now positive definite?