MATH 5343

Final Name__________________________

1. Under what conditions is the following method (to approximate $u_{tt} = u_{xx}$) stable, where $U_i^k \equiv U(x_i, t_k)$ (justify answer)? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m * dx/2)$)

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k-1} - 2U_i^{k-1} + U_{i-1}^{k-1}}{dx^2}$$

2. a. When a Galerkin finite element method is used to approximate the solution of $u'' + f(x) = 0, u(0) = u(1) = 0$, and the approximate solution is expanded as a linear combination of piecewise linear "chapeau" trial functions $\phi_i(x) : U(x) = \sum_{i=1}^{N-1} a_i \phi_i(x)$ there results a linear system $A\mathbf{a} = \mathbf{b}$ for the unknowns $\mathbf{a} = (a_1, ..., a_{N-1})$. Give formulas (involving integrals, and only first derivatives) for the elements A_{ki}, b_k of the matrix and right hand side vector.

   b. If the integrals in A_{ki} are calculated exactly, and a mid-point rule is used to approximate the integral in b_k, taking into account that $a_i = U(x_i)$, the resulting linear equations simplify (assuming uniformly spaced x_i) to:

$$\frac{-U(x_i+h) + 2U(x_i) - U(x_i-h)}{h} = \frac{h}{2}[f(x_i + h/2) + f(x_i - h/2)]$$

Find the truncation error for this "finite difference" scheme.

c. Explain why the collocation method cannot be used here in place of the Galerkin method.

3. The usual centered, second order, finite difference approximations are used to solve $u_{xx} + u_{yy} + u_{zz} = f(x, y, z)$, with $u = 0$ on the boundary of the cube $0 < x < 1, 0 < y < 1, 0 < z < 1$. If M gridlines are used in each direction, ie, $x_i = i/M, i = 0, ..., M$ and similarly for y_j, z_k, a linear system results with $(M-1)^3$ unknowns and $(M-1)^3$ equations (one equation centered on each unknown). If the unknowns are numbered by planes, that is, all unknowns on the plane z_1 are numbered first, then those on plane z_2, etc, then:

   a. Approximately how many multiplications are done if this linear system is solved by a Gauss elimination routine, which does not take any advantage of zero elements?

   b. Approximately what is the half-bandwidth of the linear system? (Half-bandwidth = max $|i - j|$ such that $A_{ij} \neq 0$.)

   c. Approximately how many multiplications are done if it is solved by a band solver, with partial pivoting?

   d. Approximately how many if a band solver is used with no pivoting?

   e. Approximately how many "divisions" are done if the SOR iterative method is used with optimal ω, assuming the number of iterations is then about cM.

4. Consider the eigenvalue problem $u_{xx} + u_{yy} = \lambda u$ with $u = 0$ on the boundary of a 2D region R. An approximate eigenfunction of the form $U(x, y) = \sum_{i=1}^{N} a_i \phi_i(x, y)$ is sought, where the ϕ_i are linearly independent and each is 0 on the boundary.

   a. If a Galerkin finite element method is used, a generalized eigenvalue problem $A\mathbf{a} = \lambda B\mathbf{a}$ results, where $\mathbf{a} = (a_1, ..., a_N)$. Give formulas (involving integrals, and only first derivatives) for the elements A_{ki}, B_{ki} of matrices A and B.

   b. Show that the matrix B in problem 4a is positive definite. (Recall that one definition of positive definite is that B is symmetric and $z^T Bz > 0$ for any nonzero vector z.) Thus, B is nonsingular.

   c. Write the inverse power method iteration for finding the smallest eigenvalue of $Az = \lambda Bz$ in a form where no inverses are calculated.

   d. If a collocation finite element method is used, with collocation points (x_k, y_k), a generalized eigenvalue problem $Aa = \lambda Ba$ again results; this time what are the elements of A_{ki}, B_{ki} of A and B? Is B now positive definite?