MATH 5343

Final Name__________________________

1. Under what conditions is the following method (to approximate $u_{tt} = u_{xx}$) stable, where $U_i^k \equiv U(x_i, t_k)$ (justify answer)? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m * dx/2)$)

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

2. a. If a Galerkin finite element method is used to approximate the solution of $u^{iv} = f(x), u(0) = u(1) = u'(1) = 0, u'(0) = A$, and the approximate solution is expanded as a linear combination of trial functions $\phi_i(x) : U(x) = \Omega(x) + \sum_{i=1}^N a_i\phi_i(x)$ where $\Omega(x)$ satisfies all four boundary conditions, and each of the ϕ_k satisfies $\phi_k(0) = \phi_k'(0) = \phi_k(1) = \phi_k'(1) = 0$, there results a linear system $A\mathbf{a} = \mathbf{b}$ for the unknowns $\mathbf{a} = (a_1, ..., a_N)$. Give formulas for the elements A_{ki}, b_k of the matrix and right hand side vector.

   b. Same question, but now the collocation method is used.

c. If the Galerkin method is used, how many continuous derivatives must the $\phi_i(x)$ have? If collocation is used? Can cubic Hermite basis functions be used for Galerkin? Collocation?

3. Suppose the ODE $u' = f(t, u)$ is approximated by:

$$11U_{n+1} - 18U_n + 9U_{n-1} - 2U_{n-2} = 6h\ f(t_{n+1}, U_{n+1})$$

a. Is the method stable? (Hint: $\lambda = 1$ is always a root of the characteristic polynomial, for consistent methods.)

b. Calculate the truncation error (Hint: don't forget to normalize).

c. Suppose this method is used to solve a linear ODE system $Au' = Bu + f(t)$, where A and B are matrices of constants (and A is nonsingular) and u is the vector of unknowns. Then every time step, a linear system must be solved which has what matrix? Suppose A and B are N by N band matrices, with half-bandwidth $L = \sqrt{N}$. Assuming N is large, the work will be what order $O(N^?)$ to solve this linear system the first time step? On every time step after the first? (Assume you take advantage of any special structure of the matrices.)

4. Write out the nonlinear overrelaxation formulas to solve the usual centered finite difference approximation to $U_{xx} + U_{yy} = U^5 + e^{x+y}$.

5. We want to find an eigenvalue of $\nabla^2 U = \lambda U$, with $U = 0$ on the boundary. If a finite element method is used, and U is expanded as a linear combination of basis function $\phi_k(x, y, z)$, each of which satisfies the boundary condition, a generalized eigenvalue problem $A\mathbf{a} = \lambda B\mathbf{a}$ will result.

   a. What are the elements A_{ki}, B_{ki} of the A and B matrices if the Galerkin method is used? Are A and/or B symmetric?

   b. Same questions, if the collocation method is used?

   c. Write out, in an efficient form, the shifted inverse power iteration for finding the eigenvalue of $A\mathbf{a} = \lambda B\mathbf{a}$ closest to p, and tell how this eigenvalue λ can be found from this iteration.