CPS 5320

Name ______________________________

1. If U is a function of $\rho = \sqrt{x^2+y^2+z^2}$ only, use the chain rule to express $U_{xx}+U_{yy}+U_{zz}$ in terms of $U_{\rho\rho}, U_\rho$ only.

   answer: $U_{\rho\rho} + 2U_\rho/\rho$

2. The 2D steady-state Navier Stokes equations, at low Reynold's number are:

$$f1 + \mu(U_{xx}+U_{yy}) = P_x$$

$$f2 + \mu(V_{xx}+V_{yy}) = P_y$$

$$U_x + V_y = 0$$

   where (U,V) is the fluid velocity vector, and μ, P are the fluid viscosity and pressure, and $(f1, f2)$ is the external force field.

   If we define a stream function $\phi(x,y)$ such that $(U,V) = (\phi_y, -\phi_x)$, show that the last (divergence) equation is automatically satisfied, and find a system of two second order equations involving ϕ and the "vorticity" $\omega \equiv U_y - V_x$.

   answer: $U_x+V_y = \phi_{yx}-\phi_{xy} = 0; \omega = \phi_{xx}+\phi_{yy}, \mu(\omega_{xx}+\omega_{yy}) = f2_x - f1_y$

3. To derive the minimal surface equation, suppose $u(x, y)$ is the surface with $u = g(x, y)$ on the boundary $\partial\Omega$ of Ω which minimizes the surface area $SA(u) \equiv \int\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\ dA$. Then if $e(x, y)$ is any smooth function with $e = 0$ on the boundary, $SA(u + \alpha e) \geq SA(u)$ for any α. From this we conclude that $f(\alpha) \equiv SA(u + \alpha e)$ has a minimum at $\alpha = 0$, and thus $f'(0) = 0$. Write out the equation $f'(0) = 0$ and explain how this equation can be used to find a partial differential equation for u. (You don't need to actually derive the partial differential equation, just outline what needs to be done.)

   answer: $\frac{df}{d\alpha}(0) = \int\int_\Omega \nabla e \bullet \left[\frac{\nabla u}{\sqrt{1+u_x^2+u_y^2}}\right]\ dA = 0$. Do an integration by parts to get $\int\int e[...]dA = 0$, since e is arbitrary, the expression in brackets must be 0.

4. The diffusion/convection/reaction equation is:

   $$C_t = \nabla \bullet [D\nabla\mathbf{C} - C\mathbf{v}] + q$$

   where $C(x, y, z, t)$ is the density of a substance, and $\mathbf{v} \equiv (U, V, W)$ is the velocity of the medium. If there is no diffusion ($D = 0$) and no reaction (source or sink) terms ($q = 0$), so only convection is operative, we get the "continuity" equation:

$$C_t + \nabla \bullet [C\mathbf{v}] = 0$$

If we follow a given point $(X(t), Y(t), Z(t))$ as it moves with the fluid (so $(X', Y', Z') = (U, V, W)$), and define the density at this moving point as $C_0(t) \equiv C(X(t), Y(t), Z(t), t)$, show, using the chain rule and the continuity equation, that

$$C_0'(t) = -(U_x + V_y + W_z)C_0(t).$$

This means that an incompressible fluid ($C_0'(t) = 0$) satisfies the divergence equation $U_x + V_y + W_z = 0$.

answer: $C_0'(t) = C_t + C_xU + C_yV + C_zW = -C_0(t)(U_x + V_y + W_z)$ since $0 = C_t + (CU)_x + (CV)_y + (CW)_z = C_t + C_xU + C_yV + C_zW + C(U_x + V_y + W_z)$

5. Chris Sewell, in his lecture on *Visualization and Data Analysis in HPC* talked about visualization with VTK, ParaView and OpenGL. Rank these three from lowest to highest level. (Lowest level means most flexible and requiring most attention to detail.)

   answer: OpenGL, VTK, ParaView

6. If the MPI Fortran program below is run on NPES=3 processors, what will be output for B, on every processor?

```
      PARAMETER (N=5)
      DOUBLE PRECISION A(N),B(N)
      INCLUDE 'mpif.h'
C                     INITIALIZE MPI
      CALL MPI_INIT (IERR)
C                     NPES = NUMBER OF PROCESSORS
      CALL MPI_COMM_SIZE (MPI_COMM_WORLD,NPES,IERR)
C                     ITASK = MY PROCESSOR NUMBER
      CALL MPI_COMM_RANK (MPI_COMM_WORLD,ITASK,IERR)
      DO I=1,N
         A(I) = 10*ITASK + I
         B(I) = A(I)
      ENDDO
C
      iroot = 0
      CALL MPI_BCAST(A,N,MPI_DOUBLE_PRECISION,iroot,
     &                 MPI_COMM_WORLD,IERR)
      CALL MPI_REDUCE(B,A,N,MPI_DOUBLE_PRECISION,
     &                 MPI_MAX,iroot,MPI_COMM_WORLD,IERR)
      CALL MPI_ALLREDUCE(A,B,N,MPI_DOUBLE_PRECISION,
     &                 MPI_SUM,MPI_COMM_WORLD,IERR)
      PRINT *, B
      CALL MPI_FINALIZE(IERR)
      STOP
      END
```

answer: 23 26 29 32 35

7. Explain why it is more efficient to distribute the columns of a matrix cyclically (0,1,2,...,NPES-1,0,1,2,...NPES-1,...) than in blocks (0,0,...0,1,1,...1,2,2,...2,...) when doing Gaussian elimination.

   answer: After first NB=N/NPES columns are zeroed, the first processor is idle the rest of the forward elimination. While the last NB columns are zeroed, all processors except the last are idle.

8. You are offered an option to buy an asset at price E at time T. If $V(t,s)$ is the value of the option at time t if the asset price is s at that time, V satisfies the Black-Scholes partial differential equation. What are the appropriate initial/boundary conditions for V?

   answer: $V(T,s) = max(s-E,0), V(t,0) = 0, \frac{\partial V}{\partial s}(t, S_{max}) = 1$

   Explanation: At the maturity time T, the value of the option is clearly $max(s-E,0)$ because if the price then is s, if $s > E$, the value of the option is $s-E$ as that is the profit one could make by immediately reselling at the going price, and if $s < E$, the value of the option is 0 because the option buyer will simply not exercise his option to buy at E. If the stock price reaches 0 at any time, it can be assumed it will never be worth E, so the option is worthless then, and the left boundary condition is $V(t,0) = 0$. If the stock price s reaches very large values, it can be assumed the price will remain above E, and so the current value of the option is $s - Ee^{-r(T-t)}$, that is, the current price minus the current value of the strike price. Hence the other boundary condition can be $\frac{\partial V}{\partial s}(t, S_{max}) = 1$.