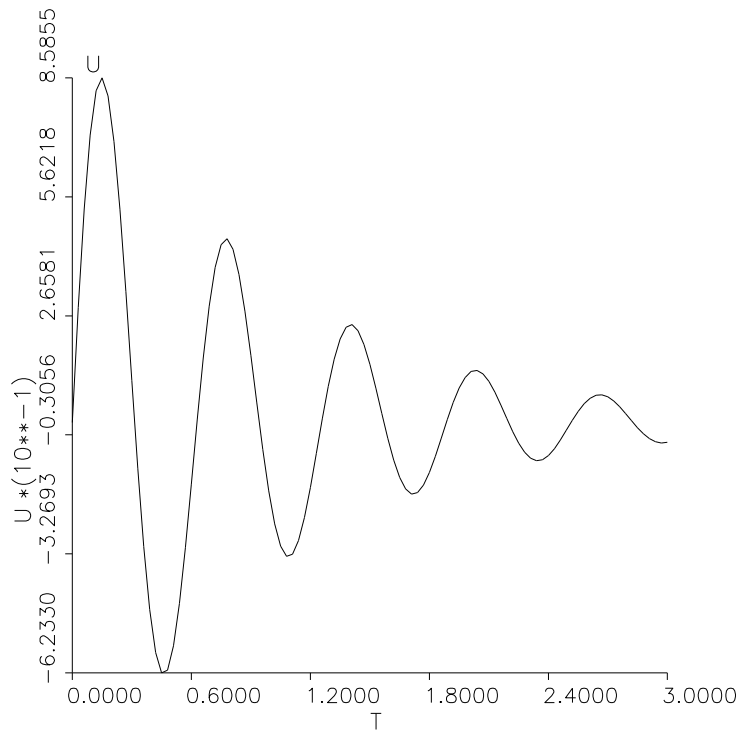


Damped Spring Problem (ODE)

0. To familiarize yourself with PDE2D, first solve the damped spring problem (1.7, p6):

$$mu'' + cu' + ku = 0, u(0) = 0, u'(0) = 10$$

where $u(t)$ is the height of the spring above its equilibrium point, so $u(0)$ and $u'(0)$ are the initial position and velocity of the spring. Take the mass $m = 1$, the spring constant $k = 101$ and the damping coefficient $c = 2$. Write this as a system of two first order ordinary differential equations, use the PDE2D GUI to generate the program, and solve out to about $t = 3$. This is a "0D" time-dependent problem, and the exact solution is $u(t) = e^{-t}\sin(10t)$.



1. a. Solve the 1D steady-state beam bending problem of Exercise 4.2.8 (p84):

$$\frac{d^2 M}{dx^2} = f(x), M(L) = 0, M'(L) = 0$$

$$\frac{d^2 u}{dx^2} = \frac{M}{EI(x)}, u(0) = 0, u'(0) = 0$$

where $u(x)$ is the beam displacement and $M(x)$ is the beam moment. Take $E = 10, I(x) = 1, L = 2.5$ (E = elastic modulus, I = moment of inertia, L = beam length). Also, put $f(x) = -1$, so there is a constant downward force, namely the weight of the uniform beam itself. The boundary conditions specify that the beam is clamped at the left end, and free and unsupported at the right end. Also solve the boundary problem analytically and compute the L_1 norm of the error, that is, the integral of $|u - u_{true}|$. Set E, I, L, f as parameters, so their values could be easily changed. Note that parameters and variables must begin with a letter in the range $A - H$ or $O - Z$, otherwise they will be integers, for example, L would be rounded down to 2, so use (for example) RL instead of L .

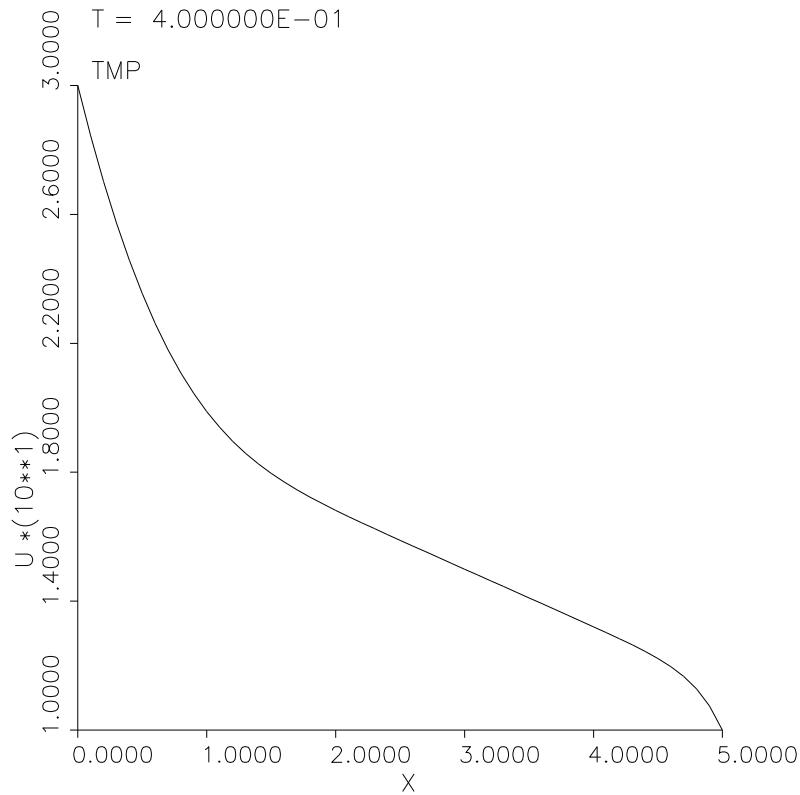
Use the PDE2D GUI (pde2d_gui [name]) which means the collocation method will be used.

- b. Repeat the problem using the Galerkin method. For this you must use the Interactive Driver (pde2d [name]). Notice that the format for the PDEs and (especially) the boundary conditions are very different than for the collocation method, so pay close attention to the documentation, especially to the hint on how to handle "mixed" (eg, one fixed, one free) boundary conditions.
2. a. Solve the 1D time-dependent heat conduction/convection problem of Example 6.1 (p108):

$$\rho C \frac{\partial T}{\partial t} + \rho C v \frac{\partial T}{\partial z} = \frac{\partial}{\partial z} \left(\kappa \frac{\partial T}{\partial z} \right) - \frac{2k}{R} (T - T_{out}), 0 \leq z \leq L$$

where $T(z, t)$ is the temperature of a fluid in a pipe of length $L = 5$, and $\rho = 1.1, C = 0.18, v = 2.0, \kappa = 0.1$ are the fluid density, specific heat, velocity and heat conductivity coefficient, respectively. The initial temperature should be $T(z, 0) = T_0 + (T_{out} - T_0)z/L$, where

$T_0 = 30$ is the temperature of the entering fluid and $T_{out} = 10$ is the temperature of the pipe wall and the outlet. Thus the boundary conditions are $T(0, t) = T_0$ and $T(L, t) = T_{out}$. Take $k/R = 0.2$, where k is the heat transfer coefficient between the fluid and the pipe wall and R is the pipe radius. Solve this forward in time to $t = 2$. Note that the spatial variable must be called x , not z , and you will not be able to call the temperature T , because T is reserved for time, so call it TMP , and use names $TMP0$, $TMPout$ instead of $T0$, $Tout$. You can use EITHER the collocation OR Galerkin method.

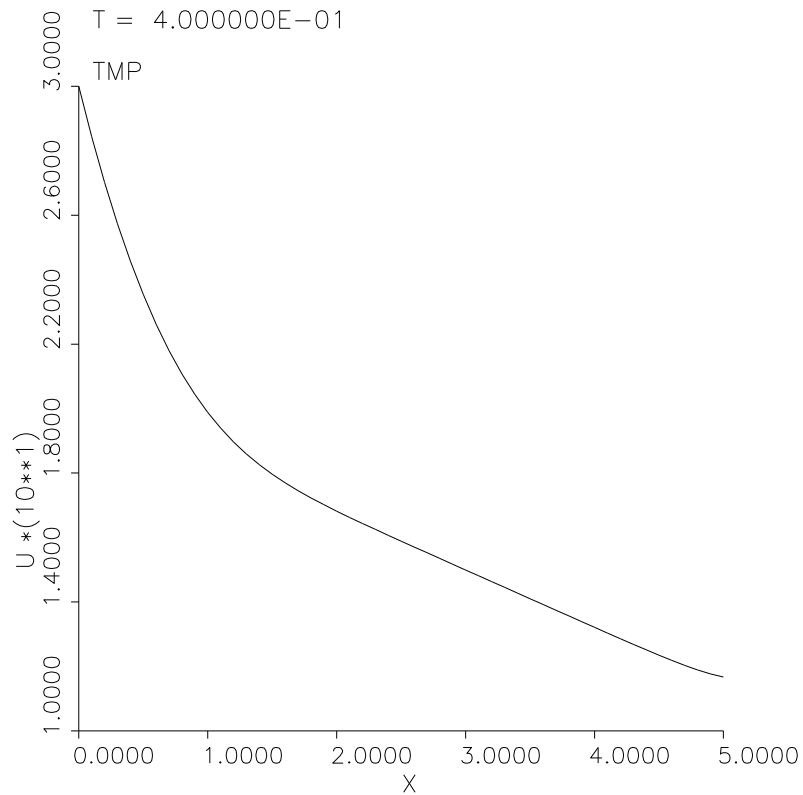


- b. Now replace the boundary condition at the outlet ($z = L$) by the non-linear radiation boundary condition (p68):

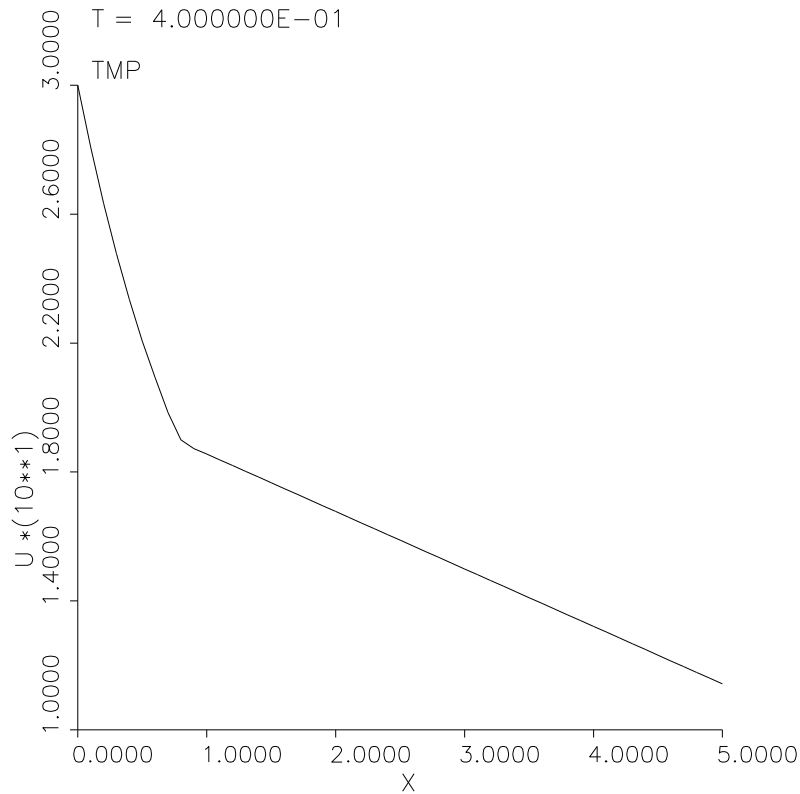
$$-\kappa \frac{dT}{dz} = \sigma(T^4 - T_{out}^4)$$

with $\sigma = 10^{-5}$, where σ is the radiation heat transfer coefficient, and repeat the problem. You will need to reset the variable `LINEAR` to

.FALSE. (collocation method only), and **FDIFF** to **.TRUE.**..



- c. Repeat problem 2a with no conduction, $\kappa = 0$. Now this is the hyperbolic transport problem of Example 8.1 (p153), and there should now be "no" boundary condition at the outlet, since it is first order in space. Note that without conduction (diffusion of heat), the discontinuity in the derivative is not smoothed out now, but simply transported forward with velocity v .



3. In mathematical finance, PDEs similar to the heat equation are often solved. For example, consider the Black-Scholes PDE, section 5.3.9, p104:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0$$

Here the solution $u(x, t)$ is the value of a European stock option, which depends on the time t and the stock price x at that time, σ is the "volatility", and r is the risk-free interest rate. The equation is similar to the backward heat equation, but it is stable because it is solved backward in time. A "call" option gives the option holder the right to buy the stock at the predetermined strike price E at the maturity time, $t = T$. At the maturity time, the value of the option is clearly $\max(x - E, 0)$ because if the price then is x , if $x > E$, the value of the option is $x - E$ as that is the profit one could make by immediately reselling at the going price, and if $x < E$, the value of the option is 0 because the option buyer will simply not exercise his option to buy at E . Thus the final condition may be written as $u(x, T) =$

$\max(x - E, 0)$, and the equation is solved backward to $t = 0$. The solution $u(x, 0)$ represents the amount one should be expected to pay for the option at a time T before maturity, if the current price is x . If the stock price reaches 0 at any time, it can be assumed it will never be worth E , so the option is worthless then, and the left boundary condition is $u(0, t) = 0$. If the stock price x reaches very large values, it can be assumed the price will remain above E , and so the current value of the option is $x - Ee^{-r(T-t)}$, that is, the current price minus the current value of the strike price. Hence the other boundary condition can be $\frac{\partial u}{\partial x}(S_{max}, t) = 1$.

This equation is solved on p117 of *Financial Engineering with Finite Elements* by Juergen Topper, John Wiley & Sons, 2005, using $E = 40$, $r = 0.1$, $\sigma = 0.2$, $S_{max} = 100$, and $T = 0.5$. The solution is found both analytically and using PDE2D there, and both give, for example, $u(39, 0) = 2.680$. Use PDE2D to solve this problem, and check your result at $x = 39$ against this value. These type of math finance problems are generally easier to set up using the collocation method, which does not require a "divergence" formulation.

