4. Consider an arch (half annulus, in 2D) described in polar coordinates as $6 < r < 10, 0 < \theta < \pi$. We want to solve the steady-state elasticity equations in this arch, that is, the equations 5.40 (p104):

$$0 = \frac{E}{2(1+\nu)}\left(U_{xx} + U_{yy} + \frac{1}{1-2\nu}(U_{xx} + V_{yx})\right) + f_1$$

$$0 = \frac{E}{2(1+\nu)}\left(V_{xx} + V_{yy} + \frac{1}{1-2\nu}(U_{xy} + V_{yy})\right) + f_2$$

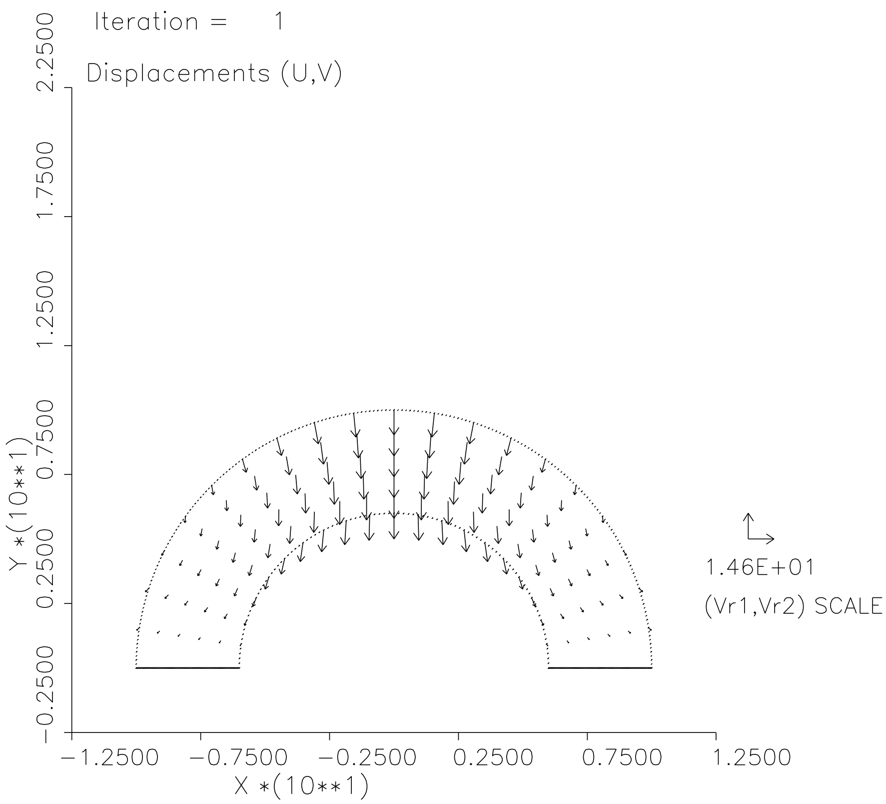
Take $E = 100$ and $\nu = 0.2$ (E = elastic modulus, $\nu$ = Poisson ratio), and take the external force vector to be $(f_1, f_2) = (0, -10)$, that is, there is a constant downward force, namely the weight of the uniform arch itself. On the two ends touching the "ground" ($\theta = 0, \pi$), the displacement vector is zero, $(U, V) = (0, 0)$. On the top and bottom of the arch ($r = 6, 10$), there are zero external forces, which means the following boundary conditions are satisfied:

$$\sigma_{xx}N_x + \sigma_{xy}N_y = g_1$$
$$\sigma_{xy}N_x + \sigma_{yy}N_y = g_2$$

where

$$\sigma_{xx} = E\frac{(1-\nu)U_x + \nu V_y}{(1+\nu)(1-2\nu)}$$
$$\sigma_{xy} = E\frac{U_y + V_x}{2(1+\nu)}$$
$$\sigma_{yy} = E\frac{\nu U_x + (1-\nu)V_y}{(1+\nu)(1-2\nu)}$$

are stresses (note: in $\sigma_{xx}$, the $xx$ does NOT indicate differentiation, it is just a subscript), $(N_x, N_y)$ is the unit outward normal to the boundary, and $(g_1, g_2)$ is the external boundary force vector, in this case $g_1 = g_2 = 0$. $N_x$ and $N_y$ are referenced in the boundary conditions as NORMx and NORMy.

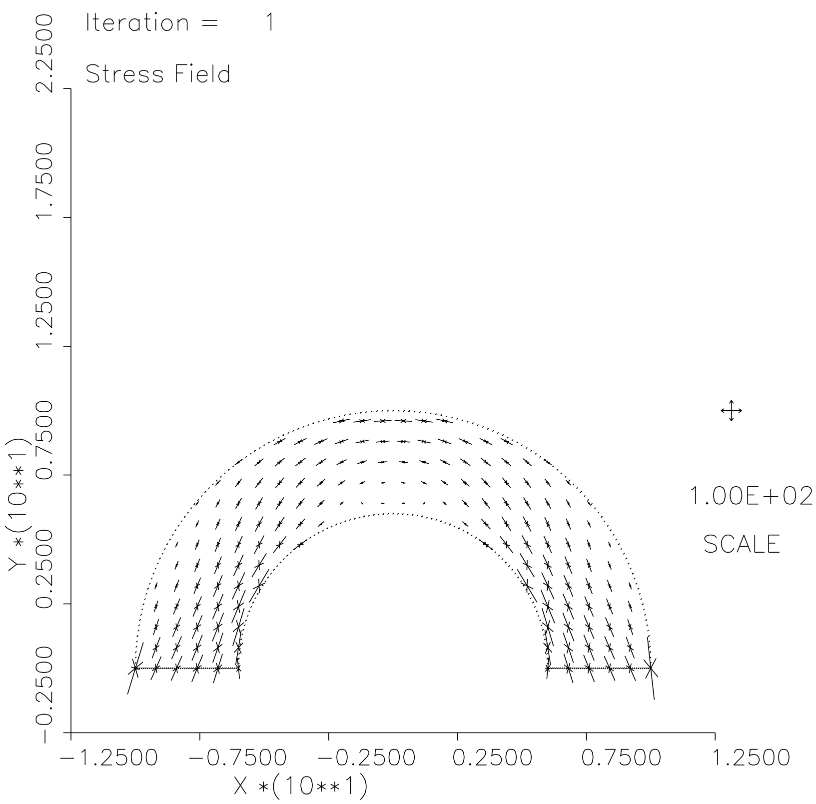



Plot the resulting displacement vector field, $(U, V)$, and calculate the integral of $V$ in the entire arch. (Note: if you use the GUI, it will generate plots of the gradients $(U_x, U_y)$ and $(V_x, V_y)$ by default, so just change IVAR1 to 1 and IVAR2 to 4 on one of these (in the Fortran) to get a plot of $(U, V)$.)

If you use the Galerkin method instead of collocation, you need to write the equations in the form:

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} + f_1 = 0$$
$$\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + f_2 = 0$$

which is equivalent to 5.40, in the 2D case. If Galerkin is used, use the initial triangulation option ITRI = 2, and note that on the free boundary, $(g_1, g_2)$ = (GB1,GB2). Stress field plots can also be made, but this is optional.




5. a. Consider the incompressible fluid flow equations 5.26 (p100):

$$\rho\left(\frac{\partial U}{\partial t} + U\,U_x + V\,U_y\right) = f1 - P_x + \mu(U_{xx} + U_{yy})$$

$$\rho\left(\frac{\partial V}{\partial t} + U\,V_x + V\,V_y\right) = f2 - P_y + \mu(V_{xx} + V_{yy})$$

$$U_x + V_y = 0$$

where $(U, V)$ is the fluid velocity vector, and $\rho, \mu, P$ are the fluid density, viscosity and pressure, and we have replaced the gravity force term in (5.26b) by a more general force vector $f = (f1, f2)$.

As noted on page 97 (for the 2D case), the fact that the divergence of the fluid velocity is zero guarantees that there is a ”stream function” $\phi$ such that $(U, V) = (\phi_y, -\phi_x)$, and the divergence equation 5.26a ($U_x + V_y = 0$) is automatically satisfied for any stream function.

Now let us define the ”vorticity” by $\omega \equiv U_y - V_x = \phi_{xx} + \phi_{yy}$. Differentiate the first equation above with respect to $y$, and the second with respect to $x$, and subtract, and show (yourself) that the pressure terms disappear, and that we are left with the equation:

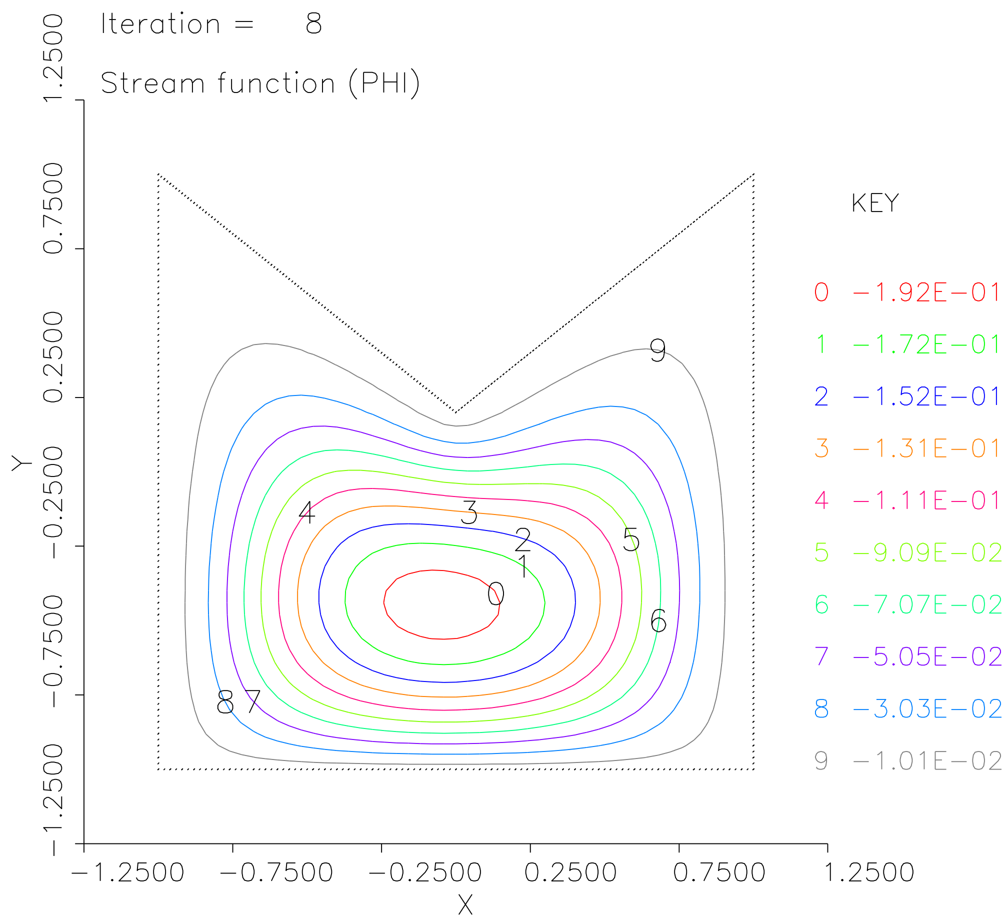
$$\rho\frac{\partial\omega}{\partial t} + \rho(\phi_y\omega_x - \phi_x\omega_y) + f2_x - f1_y = \mu(\omega_{xx} + \omega_{yy})$$

Together with $\omega = \phi_{xx} + \phi_{yy}$, we now have a system of two equations for the two unknowns $\phi$ and $\omega$.

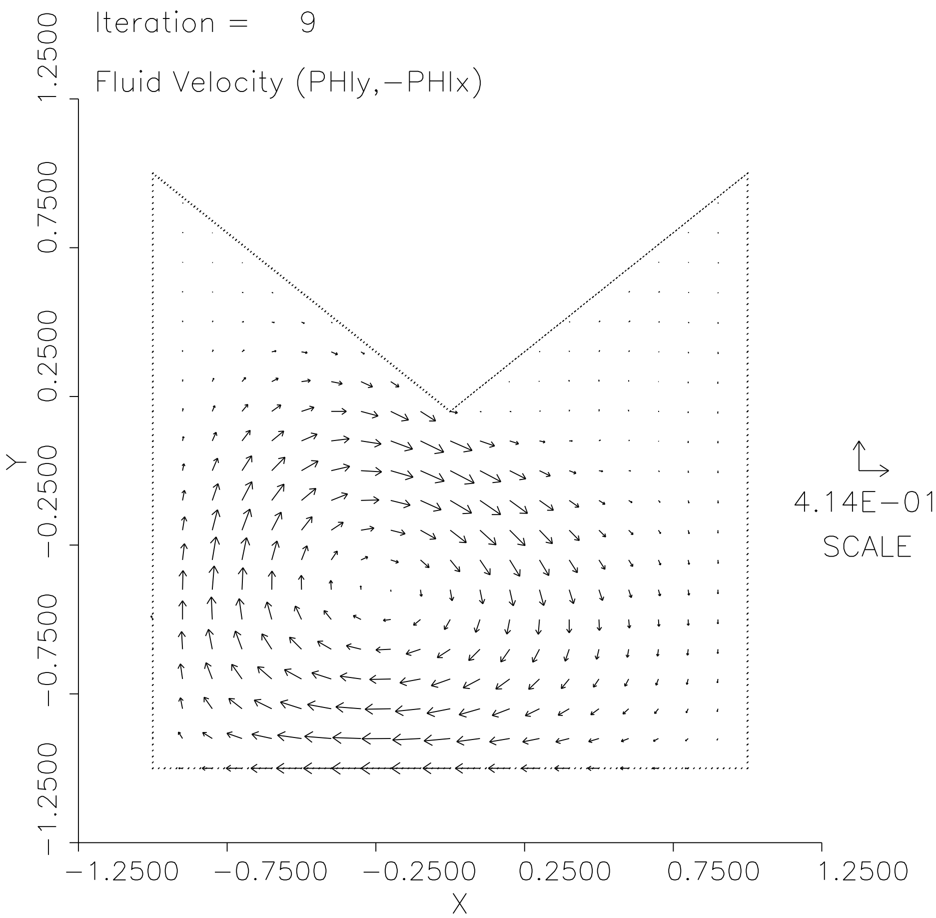
In this example we will find the steady-state flow (so the time derivative term is zero) in a pentagon with vertices at $(-1, -1), (1, -1), (1, 1), (0, 0.2), (-1, 1)$. We will assume an external force $f = (y, -x)$, which tends to rotate the fluid around the origin. On the bottom of the pentagon, we will apply ”free-slip” boundary conditions, $V = 0, U_y = 0$, and on the other four sides, we will apply ”no-slip” boundary conditions, $U = 0, V = 0$. Verify that the free-slip conditions are $\phi = 0, \omega = 0$, and that the no-slip conditions are equivalent to setting $\phi$ to $0$ (or any arbitrary constant) and its normal derivative ($\phi_x N_x + \phi_y N_y$) to $0$.

Solve this PDE problem, with $\rho = 1.1, \mu = 0.1$, and make vector plots of the fluid velocity $(\phi_y, -\phi_x)$ and a contour plot of the stream function. (Hint: set $APRINT(1) = \phi_y, BPRINT(1) = -\phi_x$ and plot (A1,B1).) Notice that the contours of the stream function are parallel to the velocity field, because the gradient of $\phi$ is normal to a level curve of $\phi$, and since $(\phi_x, \phi_y) \bullet (U, V) = 0$, it is also normal to the velocity. Also compute the integral of $\omega$. Because of the geometry, you will have to use the Galerkin method, thus you cannot use the GUI.

It should be mentioned that an alternative to this stream function approach, which still works for 3D problems, is the penalty method, in which the pressure is replaced by $P = -\alpha(U_x + V_y + W_z)$, where $\alpha$ is a large number. This equation says that a large pressure results in a small volume decrease, i.e., the fluid is “almost” incompressible.




b. Increase $\rho$ by a factor of 20 and re-solve the problem; this results in flow with a larger Reynold's number. If you have trouble getting convergence of Newton's method, you may need to multiply the nonlinear terms by $beta = min(1.d0, (T-1)/5.d0)$, which means the first iteration $(T = 1)$, you are solving a linear problem, and you are increasing the Reynold's number gradually over the next 5 iterations.


Iteration = 9
Fluid Velocity (PHIy,−PHIx)
4.14E−01
SCALE
Y
X
−1.2500
−0.7500
−0.2500
0.2500
0.7500
1.2500