

## Collocation vs Galerkin FEMs

A feature of PDE2D that is unique among commercial FEM packages is its use of the collocation method. PDE2D uses a collocation finite element method for 3D problems, and for 1D and 2D problems it offers both Galerkin and collocation FEM options. Unlike the PDE2D Galerkin method, the collocation algorithm cannot handle completely general regions, but it can solve problems in any 2D or 3D region that can be described by parametric equations with constant limits on the parameters, for example disks, spheres, cylinders, tori, cones, ellipsoids and of course, rectangles. And for all such regions, high accuracy ( $O(h^4)$ ) solutions are produced, and the PDE2D user interface is impossible to beat in terms of ease-of-use, because once the user has defined his/her parametric representation of the region (and even that is not necessary if polar, cylindrical or spherical coordinates are used), the rest of the problem description is exactly as simple as if the region were rectangular, the PDEs, boundary conditions and so on can be written in Cartesian coordinates. For example, Laplace's equation is written as  $U_{xx}+U_{yy}+U_{zz}=0$  in all 3D regions. PDE2D will also produce solution plots which reflect the true (Cartesian) geometry.

All known competitors of PDE2D use the Galerkin method only. Although the Galerkin method is easier to apply to general regions, and problems with discontinuous material properties, the collocation method is easier to apply to more general PDEs and boundary conditions:

1. The user does not have to manipulate his/her equations into the "divergence" form required by the Galerkin method, that is, the form  $(A)_x + (B)_y = F$  (for 2D), where  $A, B, F$  may be functions of the solution components and their first derivatives. While the divergence form is natural for many physical applications, it is quite unnatural for others, for example, math finance applications. And it is much easier to transform an equation "out" of divergence form than into it. For example, consider the minimal surface equation:

$$\frac{d}{dx} \left[ \frac{U_x}{\sqrt{1+U_x^2+U_y^2}} \right] + \frac{d}{dy} \left[ \frac{U_y}{\sqrt{1+U_x^2+U_y^2}} \right] = 0$$

One only has to differentiate this divergence equation to get it into a form suitable for the collocation method:

$$(1 + U_y^2)U_{xx} + (1 + U_x^2)U_{yy} - 2U_xU_yU_{xy} = 0$$

Taking the second equation and manipulating it into the divergence form required by a Galerkin method, on the other hand, is a much more formidable challenge. Of course, here we know the answer, but make any minor change to the second equation above and try to convert it back to divergence form; it will almost always be impossible to do analytically.

2. Even after a user has manipulated his/her equations into divergence form  $(A)_x + (B)_y = F$ , the Galerkin method requires "natural" boundary conditions of the

form  $An_x + Bn_y = g$ , where  $(n_x, n_y)$  is the outward unit normal to the boundary. Again, these conditions are indeed "natural" for many physical applications, but not for general PDEs. For complicated systems of PDEs, it is often very difficult or impossible to manipulate the boundary conditions into the "natural" form required by the Galerkin method.

For example, if Laplace's equation is solved,  $A = U_x, B = U_y$ , standard implementations of the Galerkin method allow only boundary conditions of the form  $U = f(x, y)$  or  $\frac{\partial U}{\partial n} = U_x n_x + U_y n_y = g(x, y, U)$ . (We are assuming a 2D problem is solved.) If one wants to solve Laplace's equation with  $U_x = 0$  on a boundary segment that is not an  $x = \text{constant}$  line, one is out of luck with Galerkin-based FEM packages. For the collocation method, this boundary condition can be simply entered as "Ux=0", regardless of the form of the PDE.