The Crank-Nicolson method approximates $u_t = Du_{xx}$ ($D > 0$) by:

$$\frac{U(x_i, t_{k+1}) - U(x_i, t_k)}{\Delta t} = \frac{D}{2}\frac{U(x_{i+1}, t_k) - 2U(x_i, t_k) + U(x_{i-1}, t_k)}{\Delta x^2} +$$
$$\frac{D}{2}\frac{U(x_{i+1}, t_{k+1}) - 2U(x_i, t_{k+1}) + U(x_{i-1}, t_{k+1})}{\Delta x^2}$$

We expand $U(x_i, t_k)$ in a Fourier series:

$$U(x_i, t_k) = \sum_m a_m(t_k)e^{Imx_i}$$

and take one term of this series and substitute for $U(x_i, t_k)$:

$$\frac{a_m(t_{k+1})e^{Imx_i} - a_m(t_k)e^{Imx_i}}{\Delta t} = \frac{D}{2}\frac{a_m(t_k)e^{Imx_{i+1}} - 2a_m(t_k)e^{Imx_i} + a_m(t_k)e^{Imx_{i-1}}}{\Delta x^2} +$$
$$\frac{D}{2}\frac{a_m(t_{k+1})e^{Imx_{i+1}} - 2a_m(t_{k+1})e^{Imx_i} + a_m(t_{k+1})e^{Imx_{i-1}}}{\Delta x^2}$$

or

$$\frac{a_m(t_{k+1}) - a_m(t_k)}{\Delta t}e^{Imx_i} = \frac{D}{2}a_m(t_k)\frac{e^{Im(x_i+\Delta x)} - 2e^{Imx_i} + e^{Im(x_i-\Delta x)}}{\Delta x^2} +$$
$$\frac{D}{2}a_m(t_{k+1})\frac{e^{Im(x_i+\Delta x)} - 2e^{Imx_i} + e^{Im(x_i-\Delta x)}}{\Delta x^2}$$

dividing through by e^{Imx_i}:

$$\frac{a_m(t_{k+1}) - a_m(t_k)}{\Delta t} = \frac{D}{2}a_m(t_k)\frac{e^{Im\Delta x} - 2 + e^{-Im\Delta x}}{\Delta x^2} + \frac{D}{2}a_m(t_{k+1})\frac{e^{Im\Delta x} - 2 + e^{-Im\Delta x}}{\Delta x^2}$$

now, using Euler's formula and a trig identity:

$$e^{Im\Delta x} - 2 + e^{-Im\Delta x} = 2cos(m\Delta x) - 2 = -4sin^2(m\Delta x/2)$$

we get:

$$a_m(t_{k+1}) - a_m(t_k) = -ra_m(t_k) - ra_m(t_{k+1})$$

where $r \equiv \frac{2D\Delta t}{\Delta x^2}sin^2(m\Delta x/2)$. Then

$$(1+r)a_m(t_{k+1}) + (-1+r)a_m(t_k) = 0$$

for fixed m, this is a first order, linear, constant coefficient, homogeneous recurrence relation, so we look for solutions of the form $a_m(t_k) = \lambda^k$, and get the characteristic polynomial:

$$(1+r)\lambda + (-1+r) = 0$$

which has the single root $\lambda = \frac{1-r}{1+r}$. Since $r \geq 0$ for all m, $|\lambda| \leq 1$ always, so for all m, $a_m(t_k) = C\lambda^k$ is bounded. So the Crank-Nicolson method is always stable.