The centered explicit method approximates $u_{tt} = c^2 u_{xx}$ by:

$$\frac{U(x_i,t_{k+1}) - 2U(x_i,t_k) + U(x_i,t_{k-1})}{\Delta t^2} = c^2 \frac{U(x_{i+1},t_k) - 2U(x_i,t_k) + U(x_{i-1},t_k)}{\Delta x^2}$$

We substitute $U(x_i,t_k) = a_m(t_k)e^{Imx_i}$:

$$\frac{a_m(t_{k+1})e^{Imx_i} - 2a_m(t_k)e^{Imx_i} + a_m(t_{k-1})e^{Imx_i}}{\Delta t^2} = c^2 \frac{a_m(t_k)e^{Imx_{i+1}} - 2a_m(t_k)e^{Imx_i} + a_m(t_k)e^{Imx_{i-1}}}{\Delta x^2}$$

or

$$\frac{a_m(t_{k+1}) - 2a_m(t_k) + a_m(t_{k-1})}{\Delta t^2}e^{Imx_i} = c^2 a_m(t_k)\frac{e^{Im(x_i+\Delta x)} - 2e^{Imx_i} + e^{Im(x_i-\Delta x)}}{\Delta x^2}$$

dividing through by e^{Imx_i}:

$$\frac{a_m(t_{k+1}) - 2a_m(t_k) + a_m(t_{k-1})}{\Delta t^2} = c^2 a_m(t_k)\frac{e^{Im\Delta x} - 2 + e^{-Im\Delta x}}{\Delta x^2}$$

now, using the same trig identities as previously, we get:

$$a_m(t_{k+1}) - 2a_m(t_k) + a_m(t_{k-1}) = c^2\frac{\Delta t^2}{\Delta x^2}a_m(t_k)[-4sin^2(m\Delta x/2)] = -4ra_m(t_k)$$

where $r \equiv c^2\frac{\Delta t^2}{\Delta x^2}sin^2(m\Delta x/2)$. Then the characteristic polynomial is:

$$\lambda^2 + (-2+4r)\lambda + 1 = 0$$

The product of the two roots of this quadratic always equals 1. If the discriminant in the quadratic formula, $(-2+4r)^2 - 4 = 16r(r-1)$ is positive, there will be two different real roots, and since their product is 1, one of them must be larger than one in absolute value, and the method is unstable. If the discriminant is negative, there will be two complex conjugate roots with the same absolute value, and their absolute values must be one, so the method is stable. Thus the method is stable if and only if $r \leq 1$ for all m, which will only happen if $\Delta t \leq \frac{\Delta x}{c}$.