Dynamic Systems, Chapter 1

1D dynamical system (first order recurrence relation) $x_{n+1} = f(x_n)$

Fixed point satisfies $x = f(x)$. Is it stable?

$x_{n+1} = f(x_n)$ and $x = f(x)$, so

$|x_{n+1} - x| = |f(x_n) - f(x)| = |f'(c)||(x_n - x)| \approx |f'(x)||(x_n - x)|$

will converge if $|f'(x)| < 1$, diverge if $|f'(x)| > 1$.

Periodic point of, say, period 3, is a fixed point of $f(f(f(x)))$, ie $x_0 = f^3(x_0)$. Call $x_1 = f(x_0), x_2 = f(x_1)$, then $x_0 = f(x_2)$. Is orbit (x_0, x_1, x_2) stable? Yes, if derivative of f^3 is less than 1 in absolute value at x_0:

$|\frac{d}{dx} f(f(f(x_0)))| = |f'(f(f(x_0)))f'(f(x_0))f'(x_0)| =$

$|f'(x_2)f'(x_1)f'(x_0)| < 1$

Example: the logistic map, $f(x) = ax(1-x), (0 < a \leq 4)$. Note that $f'(x) = a(1-2x)$.

1. Fixed points: $ax(1-x) = x$, or $x(ax - (a-1)) = 0$, which has two roots:
   a. $x = 0$. Since $f'(0) = a$, this is unstable (a ”source”), when $a > 1$.
   b. $x = (a-1)/a$. Since $f'((a-1)/a) = 2 - a$, this is stable (a ”sink”), when $1 < a < 3$.
2. Period 2 points: $f(f(x)) = x$, or

   $a(f(x))(1 - f(x)) = a[ax(1-x)][1 - ax(1-x)] = x$, or

   $a^3x^4 - 2a^3x^3 + (a^3 + a^2)x^2 + (1 - a^2)x = 0$

   Since the fixed points of f, 0 and $(a-1)/a$ must also be fixed points of $f^2(x)$, $(x - 0)$ and $(ax - (a-1))$ must be factors, and thus:

$x[ax - (a-1)][a^2x^2 - (a^2+a)x + (a+1)] = 0$

If the discriminant of the quadratic factor is positive, ie if $(a^2+a)^2 - 4a^2(a+1) = a^2(a-3)(a+1) > 0$, or $a > 3$, then there are two "new" roots:

$z_1 = [(a+1) + \sqrt{(a-3)(a+1)}]/(2a)$ and

$z_2 = [(a+1) - \sqrt{(a-3)(a+1)}]/(2a)$.

These are the period 2 points. Is this period 2 orbit stable? We calculate:

$f'(z_1) * f'(z_2) = a(1-2z_1)a(1-2z_2) = -a^2 + 2a + 4$

This quadratic polynomial has absolute value less than 1, ie, it is between -1 and 1, if a is between 3 and $1 + \sqrt{6} = 3.449$.

Thus, to summarize:

	x=0	x=(a-1)/a	(z_1, z_2)
$0 < a < 1$	stable	-	-
$1 < a < 3$	unstable	stable	-
$3 < a < 3.449$	unstable	unstable	stable
$3.449 < a < 4$	unstable	unstable	unstable