Dynamic Systems, Chapter 3

An orbit $(x_1, x_2, x_3, ...)$ is *chaotic* if:

1. It is bounded
2. It is not asymptotically periodic (or eventually periodic)
3. The Lyapunov number:
   $lim_{n\to\infty}|f'(x_1) * f'(x_2) * ...f'(x_n)|^{1/n}$
   is greater than 1.0

Condition [3] ensures sensitivity to initial conditions, since the Lyapunov number is an "average" of the derivative, and $|f(x_1) - f(y_1)| = |f'(c)| * |x_1 - y_1|$

Examples:

1. $f(x) = (x + q)_{mod\ 1}$, where $0 < q = i/j < 1$ is rational.

```
1 |    /
  |   /
  |  /
  | /
  |/
q |           /
  |          /
  |         /
  |        /
  |       /
  |      /
  |     /
  -------------
  0   1-q     1
```

Now there are no chaotic orbits, because all orbits are periodic, since $x_{j+1} = (x_1 + j * (i/j))_{mod\ 1} = x_1$.

2. $f(x) = (x + q)_{mod\ 1}$, where $0 < q < 1$ is irrational.

   Now there are no periodic orbits, but since $f'(x) = 1$ at all points except the single point $x = 1 - q$, the Lyapunov number is 1 for all orbits, so there is no sensitivity to initial conditions. If x_1 and y_1 are very close, their orbits will stay close, in fact they will neither separate nor converge.

3. $f(x) = (2x)_{mod\ 1}$.

   If $x = 0.b_1b_2b_3...$ is the binary representation for x, $f(x)$ is found by simply shifting the "decimal" point one bit to the right (multiplying by 2) and truncating any integer part (taking mod 1 of the result). If x_1 has a repeating or truncating binary representation, that is, if x_1 is rational, then the orbit $(x_1, x_2, x_3, ...)$ is eventually periodic, so this orbit is not chaotic. But if x_1 is irrational, the orbit $(x_1, x_2, x_3, ...)$ is chaotic, because it is not asymptotically periodic, and the Lyapunov number is 2, because $f'(x) = 2$ everywhere except $x = 0, 0.5, 1.0$. That means there is sensitive dependence on initial conditions: if x_1 and y_1 are very close, say, they differ beginning in the 100^{th} binary bit, their separation will approximately double each iteration, at least for the first 100 iterations.

4. $T(x) = 2x$ when $x < 1/2$, and
   $T(x) = 2(1 - x)$ when $x \geq 1/2$.

   This is the "tent map". Again we can define the map in terms of what it does to the binary representation of x: If the first bit is a $0, T$ simply shifts the decimal point to the right one bit (multiplies by 2). If the first bit is a $1, T$ switches all $0s$ and $1s$ (subtracts x from $1 = 0.1111111...$), and then shifts the decimal point to the right one bit (multiplies by 2). So, like the previous map, if x_1 is irrational, the orbit is not eventually or asymptotically periodic, and its Lyapunov number is 2 because $|T'(x)| = 2$ everywhere except three points. Thus $(x_1, x_2, x_3, ...)$ is a chaotic orbit, if x_1 is irrational. If x_1 is rational, $(x_1, x_2, x_3, ...)$ is eventually periodic. To see this, suppose the binary representation of x_1 has a repeating block of M bits. Then, for i greater than some i_0, either $x_{i+M} = x_i$ or $x_{i+M} = 1 - x_i$, depending on whether an even or odd number of bit reflections are done during the M iterations. In the first case, the period is M; in the second case, do another M iterations and then either $x_{i+2M} = x_{i+M}$ or $x_{i+2M} = 1 - x_{i+M} = x_i$. So the period is $2M$ or less. (With a little more work, it can be shown that the period is actually M or less.)

5. $f(x) = 4x(1 - x)$

   This is the logistic map with $a = 4$. Now this map is conjugate to the tent map $T(x)$, with conjugacy $C(x) = \frac{1}{2}[1 - cos(\pi x)]$, because $C(T(x)) = f(C(x)) = sin^2(\pi x)$ as can be directly verified. This means for every orbit $(x_1, x_2, x_3, ...)$ of the tent map, there is a corresponding orbit $(C(x_1), C(x_2), C(x_3), ...)$ of the logistic map, because $C(x_{n+1}) = C(T(x_n)) = f(C(x_n))$. Clearly, if $(x_1, x_2, x_3, ...)$ is bounded, not asymptotically periodic and depends sensitively on the initial point, the same is true of $(C(x_1), C(x_2), C(x_3), ...)$. Thus there are also an infinite number of chaotic orbits for the logistic map (with $a = 4$), one starting with $C(x_1)$ for any irrational x_1. All orbits starting with $C(x_1)$ for x_1 rational produce periodic orbits (all repelling!); however, ”most” orbits are chaotic.