

Dynamic Systems, Chapter 3

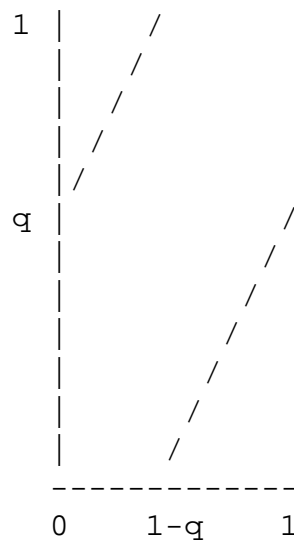
An orbit (x_1, x_2, x_3, \dots) is *chaotic* if:

1. It is bounded
2. It is not asymptotically periodic (or eventually periodic)
3. The Lyapunov number:
 $\lim_{n \rightarrow \infty} |f'(x_1) * f'(x_2) * \dots * f'(x_n)|^{1/n}$
 is greater than 1.0

Condition [3] ensures sensitivity to initial conditions, since the Lyapunov number is an "average" of the derivative, and $|f(x_1) - f(y_1)| = |f'(c)| * |x_1 - y_1|$

Examples:

1. $f(x) = (x + q)_{mod 1}$, where $0 < q = i/j < 1$ is rational.



Now there are no chaotic orbits, because all orbits are periodic, since $x_{j+1} = (x_1 + j * (i/j))_{mod 1} = x_1$.

2. $f(x) = (x + q)_{\text{mod } 1}$, where $0 < q < 1$ is irrational.

Now there are no periodic orbits, but since $f'(x) = 1$ at all points except the single point $x = 1 - q$, the Lyapunov number is 1 for all orbits, so there is no sensitivity to initial conditions. If x_1 and y_1 are very close, their orbits will stay close, in fact they will neither separate nor converge.

3. $f(x) = (2x)_{\text{mod } 1}$.

If $x = 0.b_1b_2b_3\dots$ is the binary representation for x , $f(x)$ is found by simply shifting the "decimal" point one bit to the right (multiplying by 2) and truncating any integer part (taking mod 1 of the result). If x_1 has a repeating or truncating binary representation, that is, if x_1 is rational, then the orbit (x_1, x_2, x_3, \dots) is eventually periodic, so this orbit is not chaotic. But if x_1 is irrational, the orbit (x_1, x_2, x_3, \dots) is chaotic, because it is not asymptotically periodic, and the Lyapunov number is 2, because $f'(x) = 2$ everywhere except $x = 0, 0.5, 1.0$. That means there is sensitive dependence on initial conditions: if x_1 and y_1 are very close, say, they differ beginning in the 100^{th} binary bit, their separation will approximately double each iteration, at least for the first 100 iterations.

4. $T(x) = 2x$ when $x < 1/2$, and
 $T(x) = 2(1 - x)$ when $x \geq 1/2$.

This is the "tent map". Again we can define the map in terms of what it does to the binary representation of x : If the first bit is a 0, T simply shifts the decimal point to the right one bit (multiplies by 2). If the first bit is a 1, T switches all 0s and 1s (subtracts x from $1 = 0.111111\dots$), and then shifts the decimal point to the right one bit (multiplies by 2). So, like the previous map, if x_1 is irrational, the orbit is not eventually or asymptotically periodic, and its Lyapunov number is 2 because $|T'(x)| = 2$ everywhere except three points. Thus (x_1, x_2, x_3, \dots) is a chaotic orbit, if x_1 is irrational. If x_1 is rational, (x_1, x_2, x_3, \dots) is eventually periodic. To see this, suppose the binary representation of x_1 has a repeating block of M bits. Then, for i greater than some i_0 , either $x_{i+M} = x_i$ or $x_{i+M} = 1 - x_i$, depending on whether an even or odd number of bit reflections are done during the M iterations. In the first case, the period is M ; in the second case, do another M iterations and then either $x_{i+2M} = x_{i+M}$ or $x_{i+2M} = 1 - x_{i+M} = x_i$. So

the period is $2M$ or less. (With a little more work, it can be shown that the period is actually M or less.)

5. $f(x) = 4x(1 - x)$

This is the logistic map with $a = 4$. Now this map is conjugate to the tent map $T(x)$, with conjugacy $C(x) = \frac{1}{2}[1 - \cos(\pi x)]$, because $C(T(x)) = f(C(x)) = \sin^2(\pi x)$ as can be directly verified. This means for every orbit (x_1, x_2, x_3, \dots) of the tent map, there is a corresponding orbit $(C(x_1), C(x_2), C(x_3), \dots)$ of the logistic map, because $C(x_{n+1}) = C(T(x_n)) = f(C(x_n))$. Clearly, if (x_1, x_2, x_3, \dots) is bounded, not asymptotically periodic and depends sensitively on the initial point, the same is true of $(C(x_1), C(x_2), C(x_3), \dots)$. Thus there are also an infinite number of chaotic orbits for the logistic map (with $a = 4$), one starting with $C(x_1)$ for any irrational x_1 . All orbits starting with $C(x_1)$ for x_1 rational produce periodic orbits (all repelling!); however, "most" orbits are chaotic.