Dynamic Systems, Chapter 4

THE TENT MAP

Consider the "tent" map on page 162: $x_{n+1} = f(x_n)$, where:

$$\begin{array}{rcll} f(x) & = & ax, & -\infty < x < \frac{1}{2} \\ f(x) & = & a(1-x), & \frac{1}{2} < x < \infty \end{array}$$

we will consider here the case $a = 3$. Note that:

1. If $x < 0, f(x) = 3x$, so all points less than 0 go to $-\infty$.
2. If $x > 1, f(x) < 0$, so all points greater than 1 also go to $-\infty$.
3. If $\frac{1}{3} < x < \frac{2}{3}, f(x) > 1$ so all points in the middle third of the unit interval also end up at $-\infty$.
4. If $0 < x < \frac{1}{3}, x$ has a base 3 representation $x = (0.0t_2t_3t_4...)_3$, so $f(x) = 3x = (0.t_2t_3t_4...)_3$.
5. If $\frac{2}{3} < x < 1, x$ has a base 3 representation $x = (0.2t_2t_3t_4...)_3$, so $f(x) = 3(1-x) = 3 * (0.2222222_3 - (0.2t_2t_3t_4...)_3) = 3 * (0.0t'_2t'_3t'_4...)_3 = (0.t'_2t'_3t'_4...)_3$, where the prime means 0s and 2s are switched $(t'_i = 2 - t_i)$.

THE CANTOR SET

Thus, the first iteration, everything in the middle third $(x = (0.1???)_3)$ goes out of the unit interval, never to return. On the second iteration, everything in the middle third of the first third $(x = (0.01???)_3)$ and in the middle third of the last third $(x = (0.21???)_3)$ are kicked out, and so on.

So what values of x do NOT ever leave the unit interval? We throw out the middle third, then the middle thirds of the remaining two sections, then the middle thirds of the remaining 4 sections, etc.; are there any points that do not eventually get thrown out? Yes, any x whose base 3 representation does not have any 1s in it will be left, if x has a 1 in the n-th base 3 digit it will exit after n iterations. The set of points that do not belong to the "basin" of infinity is called the Cantor set, shown on page 150.

The Cantor set is a ”fractal” set–it looks basically the same no matter how far you ”zoom” in and magnify it. What is the ”size” of the Cantor set? In one sense, there are as many points in the Cantor set as in the entire unit interval, because you can take any point in the unit interval, write its binary representation, change all 1s to 2s and you have the base three representation of a point in the Cantor set. Yet in another sense the Cantor set is much smaller than the unit interval. If you throw out $\frac{1}{3}$ of the unit interval, then $\frac{1}{3}$ of what's left, and so on, each iteration you have left only $\frac{2}{3}$ of what was left the previous iteration, so after m iterations you have only $(\frac{2}{3})^m$ left, and that goes to 0.

BOX DIMENSION

We define the ”box dimension” of a set as follows: if it requires N boxes of edge ϵ to cover the set, and if $N \approx c * (1/\epsilon)^d$, we say d is the box dimension of the set. For example, a single point requires only one interval no matter how small ϵ is, so $N = (1/\epsilon)^0$ and a point has dimension 0. The entire unit interval can be covered with $N = 1/\epsilon$ intervals of edge ϵ, so $N \approx (1/\epsilon)^1$, and the unit interval has dimension $d = 1$. The unit square in R^2 can be covered with $N = (1/\epsilon)^2$ squares of edge ϵ, so $d = 2$. If $N \approx c * (1/\epsilon)^d$, we can solve for d by taking the logarithm of both sides:

$$ln(N) \approx ln(c) + d * ln(1/\epsilon)$$

$$\frac{ln(N) - ln(c)}{ln(1/\epsilon)} \approx d$$

when N is large and ϵ is small, $ln(c)$ is negligible, and in the limit,

$$\frac{ln(N)}{ln(1/\epsilon)} \approx d$$

For the Cantor set, suppose we throw out the middle third m times, then there are $N = 2^m$ subintervals left, each of size $\epsilon = (1/3)^m$. Since the Cantor set is a subset of what is left, the entire Cantor set can be covered by $N = 2^m$ intervals of length $\epsilon = (1/3)^m$, so

$$d \approx \frac{ln(N)}{ln(1/\epsilon)} = \frac{ln(2^m)}{ln(3^m)} = \frac{m*ln(2)}{m*ln(3)} = \frac{ln(2)}{ln(3)} = 0.631$$

so we can say the dimension of the Cantor set is the fraction 0.631.

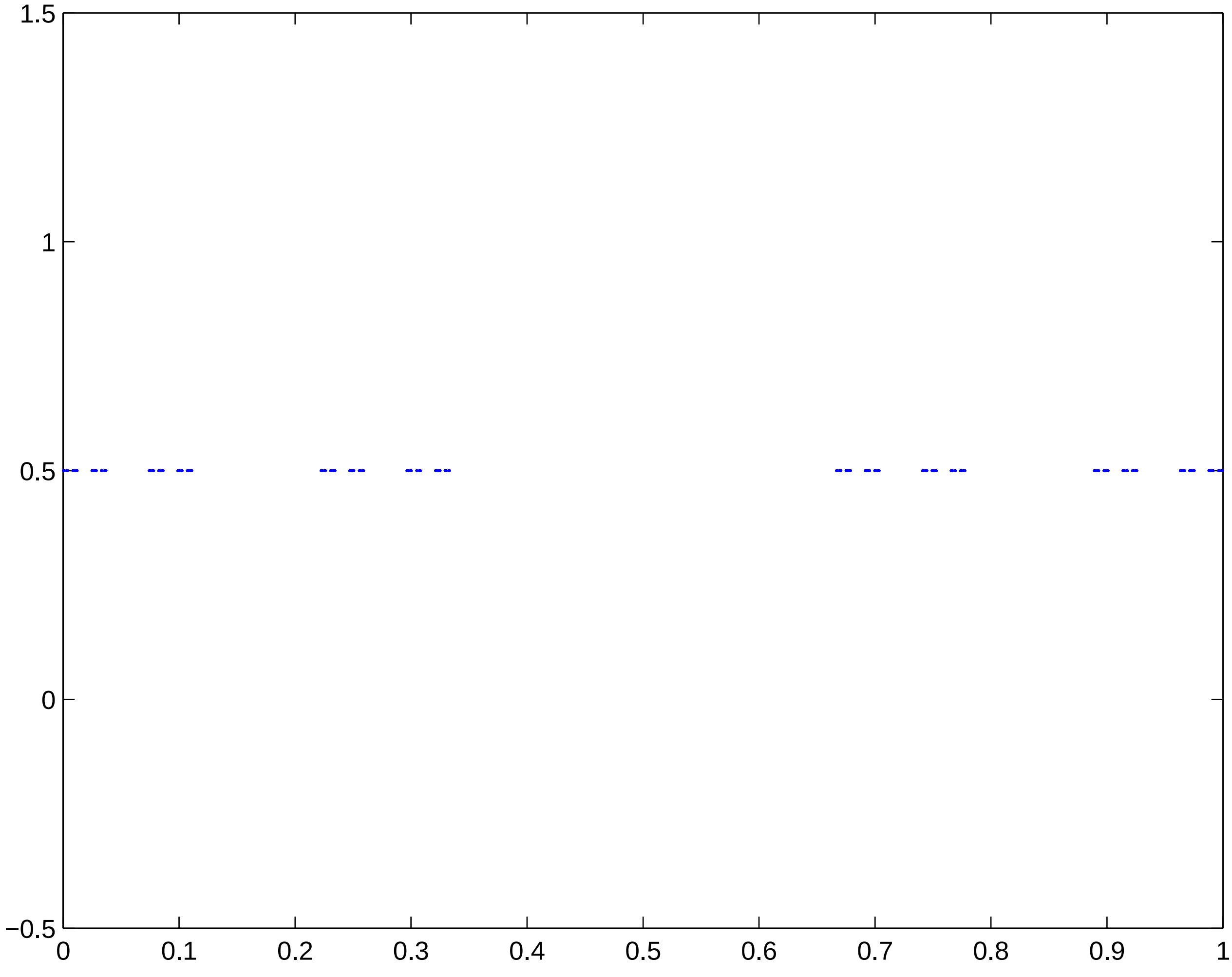
Now consider the Sierpinski gasket set, shown on page 159. After m iterations, we have $N = 3^m$ triangles of edge $(1/2)^m$, which can be covered with squares of edge $\epsilon = (1/2)^m$, so

$$d \approx \frac{ln(N)}{ln(1/\epsilon)} = \frac{ln(3^m)}{ln(2^m)} = \frac{m*ln(3)}{m*ln(2)} = \frac{ln(3)}{ln(2)} = 1.585$$

ITERATED FUNCTION SYSTEMS

Let's define a "non-deterministic" map $f(x)$ to be $f_1(x) = x/3$ with probability 0.5, and $f_2(x) = 2/3 + x/3$ with probability 0.5. That means, each iteration, we flip a coin and either (if heads) map the entire unit interval linearly onto the left third, or (if tails) map the entire unit interval onto the right third of the unit interval. Since $f(x)$ may change each iteration, this does not really represent a dynamical system, it is basically just a tool for generating a plot of the Cantor set. Now if x_n has the base 3 representation $0.abc..._3$, each iteration we either (if $f(x) = f_1(x)$) shift all digits to the right one digit and insert a leading 0, that is, $x_{n+1} = 0.0abc..._3$, or (if $f(x) = f_2(x)$) shift all digits to the right one digit and insert a leading 2, that is, $x_{n+1} = 0.2abc..._3$. Now if we plot all points x_n beginning with, say, $n = 10$, these points will all have base 3 representations whose first 10 digits are all $0s$ or $2s$, thus, they will all be in the Cantor set (as far as the naked eye can determine, anyway). And they will be more or less randomly distributed over the Cantor set, because we keep inserting random $0s$ or $2s$ in the first digit after the "decimal" point, as we shift the other digits to the right. Below is a plot of points x_{10} through x_{1000} using this iterated function technique.

For Computer Experiment 4.2, you are to generate plots of the Sierpinski gasket and the Sierpinski carpet using this iterated function technique. In exercise T4.3a, you define 3 maps, one of which maps the entire triangle onto the lower left quarter triangle, one maps onto the upper quarter triangle, and one maps onto the lower right quarter triangle. To generate a plot of the Sierpinski gasket, you randomly select one of these three maps each iteration, and plot all points, starting with say the $10th$ point. In exercise T4.4a, you repeat T4.3a for the Sierpinski carpet, defining the four maps which will be used in Computer Experiment 4.2 to also plot the carpet.



Cantor set