Dynamic Systems, Chapter 7

Autonomous system of differential equations:

$$\begin{aligned} x_1'(t) &= f_1(x_1, ..., x_n) \\ . &= . \\ x_n'(t) &= f_n(x_1, ..., x_n) \end{aligned}$$

or $x'(t) = f(x)$.

Linear autonomous system:

$$\begin{aligned} x_1' &= a_{11}x_1 + ... + a_{1n}x_n - b_1 \\ . &= . \\ x_n' &= a_{n1}x_1 + ... + a_{nn}x_n - b_n \end{aligned}$$

or $x' = Ax - b$.

Equilibrium points are points where $f(x) = 0$, or $Ax - b = 0$, in the linear case. But equilibrium points, like the fixed points of discrete dynamic systems, may be stable or unstable.

Suppose $Ax^* = b$, so that x^* is an equilibrium point, and suppose A is ”diagonalizable”, that is, $A = S^{-1}DS$, where D is a diagonal matrix, containing the eigenvalues $d_{ii} = \lambda_i$ of A. Then:

$$x' = Ax - b = Ax - Ax^* = A(x - x^*) = S^{-1}DS(x - x^*)$$

or

$$Sx' = DS(x - x^*).$$

and define $z(t) = S(x(t) - x^*)$, so $z' = Sx'$. Then $z' = Dz$ or,

$$\begin{aligned} z_1' &= \lambda_1 z_1 \\ . &= . \\ z_n' &= \lambda_n z_n \end{aligned}$$

and so $z_k = c_k e^{\lambda_k t}$. Although we assume A and b are real, the eigenvalue $\lambda_k = p_k + iq_k$ may be complex, so:

$z_k = c_k e^{\lambda_k t} = c_k e^{p_k t} e^{i q_k t} = c_k e^{p_k t}(cos(q_k t) + i * sin(q_k t))$

Now every z_k converges to 0, ie, $z(t) = S(x(t) - x^*)$ converges to 0, which means $x(t)$ converges to the equilibrium point x^*, if the real part p_k of every eigenvalue is negative. This statement is still true even if A is not diagonalizable, but the proof is more difficult. Thus if all eigenvalues of A have negative real parts, the equilibrium point x^* is globally stable. If any eigenvalues have positive real parts, the equilibrium point is unstable. If there are eigenvalues with zero real parts, but none with positive real parts, further analysis is needed.

An equilibrium point x^* of a nonlinear system is a point where $f(x^*) = 0$. In the nonlinear case, we write:

$x' = f(x) = f(x) - f(x^*) \approx Df(x^*)(x - x^*)$

so we can, in the neighborhood of an equilibrium point, approximate the nonlinear system by a linear system, with A replaced by the Jacobian of f at x^*. Now the eigenvalues of the Jacobian $Df(x^*)$ determine stability, though one cannot say anything about global stability, only local stability.

To relate stability for discrete systems to stability for differential equation systems, we can approximate the linear differential equation system $x' = Ax - b = A(x - x^*)$ using Euler's method:

$\frac{x^{n+1} - x^n}{h} = A(x^n - x^*)$, or

$x^{n+1} = x^n + hA(x^n - x^*)$, or

$x^{n+1} - x^* = x^n - x^* + hA(x^n - x^*)$

If we define $e^n = x^n - x^*$, then:

$e^{n+1} = (I + hA)e^n$

and from chapter 2 we know that an equilibrium point x^* will be stable if all eigenvalues of $I + hA$ are less than one is absolute value. If the eigenvalues of A are $\lambda_k = p_k + iq_k$, the eigenvalues of $I + hA$ are $\mu_k = 1 + h(p_k + iq_k)$, and

$|\mu_k|^2 = (1 + hp_k)^2 + (hq_k)^2 = 1 + 2hp_k + h^2(p_k^2 + q_k^2)$

when h is small, so that the Euler equation approximates the differential equation system, we have:

$$|\mu_k|^2 \approx 1 + 2hp_k$$

which will be less than 1 when p_k is negative.

Nonlinear Example

Consider the Example 7.19 on page 299 of your text:

$$\begin{aligned} x' &= x^2 - y^2 \\ y' &= xy - 4 \end{aligned}$$

When we set $x^2 - y^2 = 0$ and $xy - 4 = 0$ we find two equilibrium points, $(2, 2)$ and $(-2, -2)$.

The Jacobian matrix is:

$$\begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}$$

and the eigenvalues are the roots of $(2x - \lambda)(x - \lambda) + 2y^2 = 0$, which are:

$$\lambda = \frac{1}{2}(3x \pm \sqrt{x^2 - 8y^2})$$

At the equilibrium point $(2, 2)$, then, the eigenvalues are $\lambda = 3 \pm \sqrt{7}i$, and since the real parts of both eigenvalues are positive, this point is unstable. At the other equilibriuim point $(-2, -2)$, the eigenvalues are $\lambda = -3 \pm \sqrt{7}i$, and this point is stable.

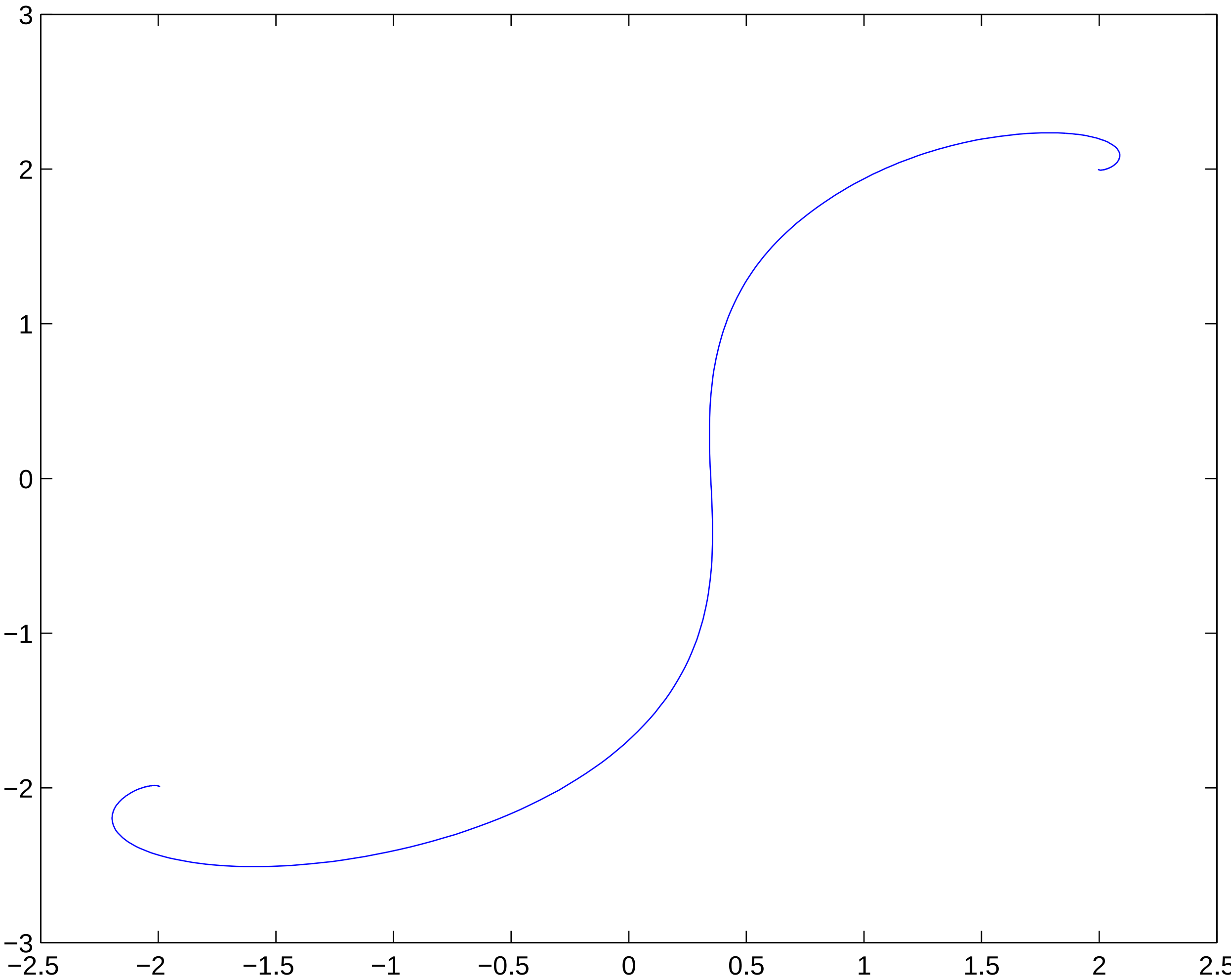
We use MATLAB to solve this system:

```
tspan = 0:0.01:5;
x0 = [1.998 1.999];
[T X] = ode45('f719',tspan,x0);
```

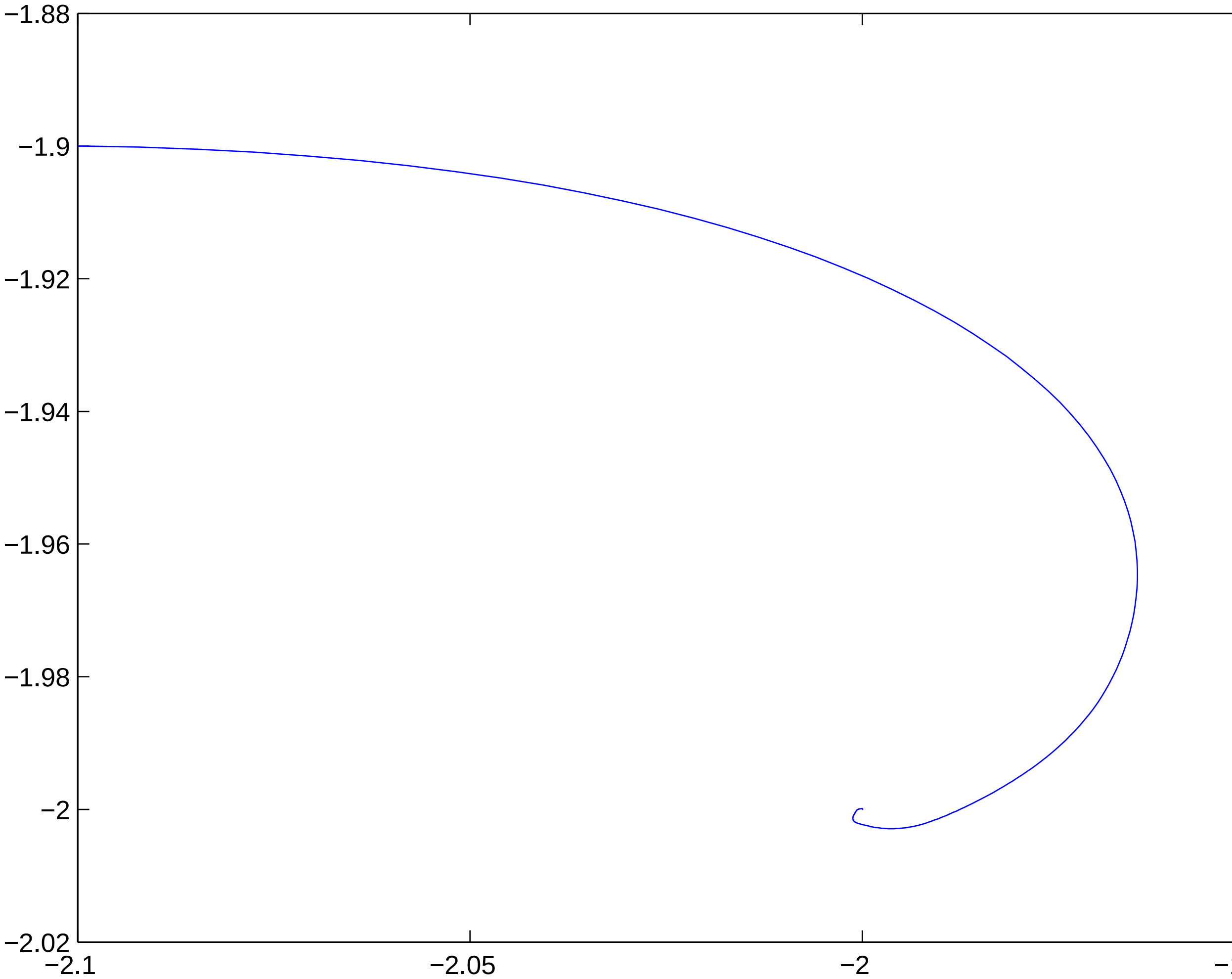
```
plot(X(:,1),X(:,2))
x0 = [-2.1 -1.9];
[T X] = ode45('f719',tspan,x0);
figure
plot(X(:,1),X(:,2))

function xp = f719(t,x)
xp = zeros(2,1);
xp(1) = x(1)^2-x(2)^2;
xp(2) = x(1)*x(2) - 4;
```

first, starting from a point $(1.998, 1.999)$ very close to the unstable equilibrium point $(2, 2)$, then, starting from a point close to the stable equilibrium point $(-2, -2)$.



Starting At (1.998,1.999)



Starting At (-2.1,-1.9)

Oscillators

Consider the differential equation:

$$m\frac{d^2x}{dt^2} = f(x) - a\frac{dx}{dt}$$

or, written as a system of two first order equations:

$$x_1' = x_2$$
$$mx_2' = f(x_1) - ax_2$$

where $f(x) = -P'(x)$. This represents an oscillator of mass m, in a potential energy field given by $P(x)$ ($f(x) = -P'(x)$ is the force field, the force acts in the direction of decreasing potential energy), with a frictional (damping) force proportional to the velocity, and in the direction opposite the velocity. For an oscillating spring, $f(x) = -kx$, for a pendulum, $f(x) = -k * sin(x)$, and for the Duffing oscillator, $f(x) = x - x^3$.

The total energy of the oscillator is given by $E(t) = \frac{1}{2}m(x')^2 + P(x)$, where the first term is the kinetic energy, the second is the potential energy. Now

$$\frac{dE}{dt} = mx'x'' + P'(x)x' = x'(mx'' - f(x)) = -a(x')^2$$

so if $a = 0$ (undamped oscillator), the energy is constant, if $a > 0$ (damped oscillator) the energy decreases until the velocity is 0.

An undamped oscillator, such as a spring or pendulum, can exhibit periodic motion. A damped oscillator cannot, it will rather converge to an equilibrium point, where the velocity is zero (kinetic energy is zero) and the potential energy has at least a local minimum (eg, computer problem 8.2 for the Duffing oscillator). However, if you add an external forcing term, $g(t)$, (eg, computer problem 7.3 for the Duffing oscillator) now fresh energy is continually pumped into the system, and then even a damped oscillator can exhibit periodic behavior. But the period of the oscillator in this case must obviously be a multiple of the period of the external force (2π in the case of computer problem 7.3).

Computer problem 7.2 can be written as a second order equation:

$$x'' = -x + (x')^2$$

which can be thought of as a spring oscillator with nonlinear damping, where the damping coefficient $a = -x'$ is not constant. In fact, now a is not always positive, when the oscillator is on its way up $(x' > 0)$, the motion is accelerated, on its way down the motion is damped, and it is not clear what one should expect, periodic motion is not ruled out.

The situation in computer problem 8.1 (Van der Pol equation):

$$x'' = -x + (1 - x^2)x'$$

is similar, now $a = x^2 - 1$, so again sometimes there may be damping, sometimes acceleration of the motion. In fact, there is damping when the oscillator is far from $x = 0$, and acceleration when it is close to 0.