Dynamic Systems, Chapters 8,9

What kind of orbits are possible with autonomous systems of differential equations?

1. Linear 2D systems. Examples of possible limit sets:

   a. An isolated equilibrium point. If $\alpha, \beta < 0$, the equilibrium point $(0,0)$ is globally attracting:

   $$\begin{aligned} x'(t) &= \alpha x \\ y'(t) &= \beta y \end{aligned}$$

   Solution is

   $$\begin{aligned} x(t) &= x_0 e^{\alpha t} \\ y(t) &= y_0 e^{\beta t} \end{aligned}$$

   Note: in the more general linear case, $v' = Av$, if $A = S^{-1}DS$ is diagonalizable, the system can be written $w' = Dw$, where $w = Sv$, so such problems are all equivalent to the above simple example, with (x, y) replaced by (w_1, w_2) and α, β replaced by the eigenvalues of A.

   b. Unbounded orbits. If $\alpha > 0$ or $\beta > 0$, the above system has unbounded orbits. Note that $(0,0)$ is still an equilibrium point, but is unstable.

   c. Non-isolated equilibrium points. If $\alpha < 0, \beta = 0$, the above system will converge to $(0, y_0)$. So every point on the line $x = 0$ is an equilibrium point.

   d. Periodic orbits.

   $$\begin{aligned} x'(t) &= y \\ y'(t) &= -x \end{aligned}$$

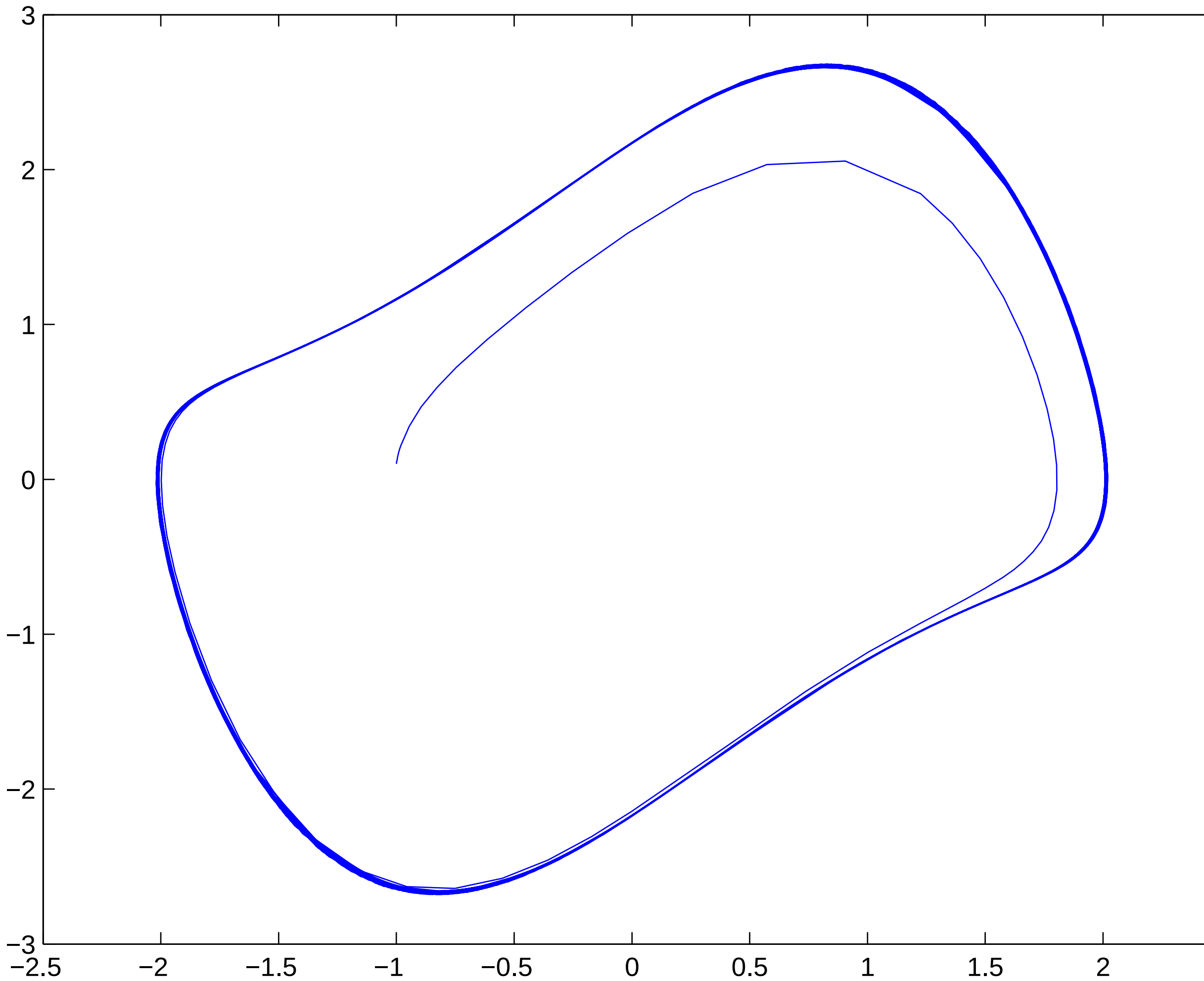
   Solution is

   $$\begin{aligned} x(t) &= x_0 cos(t) + y_0 sin(t) &= r_0 cos(\theta_0 - t) \\ y(t) &= -x_0 sin(t) + y_0 cos(t) &= r_0 sin(\theta_0 - t) \end{aligned}$$

   Note: no chaotic orbits.

2. Nonlinear 2D systems. Unlike linear case, now a periodic limit set is possible, for example, in the Van der Pol equation:

$$\begin{aligned} x'(t) &= v \\ v'(t) &= (1-x^2)v - x \end{aligned}$$

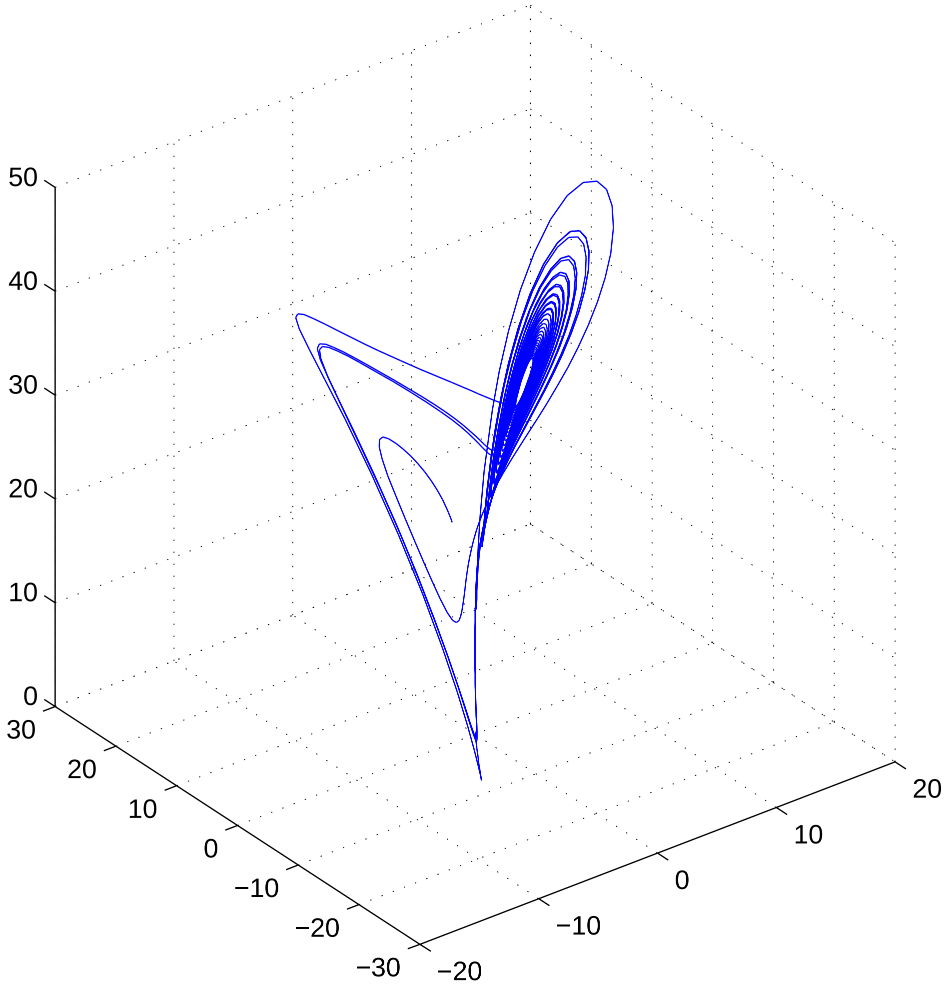


Attracting periodic limit for Van der Pol equation

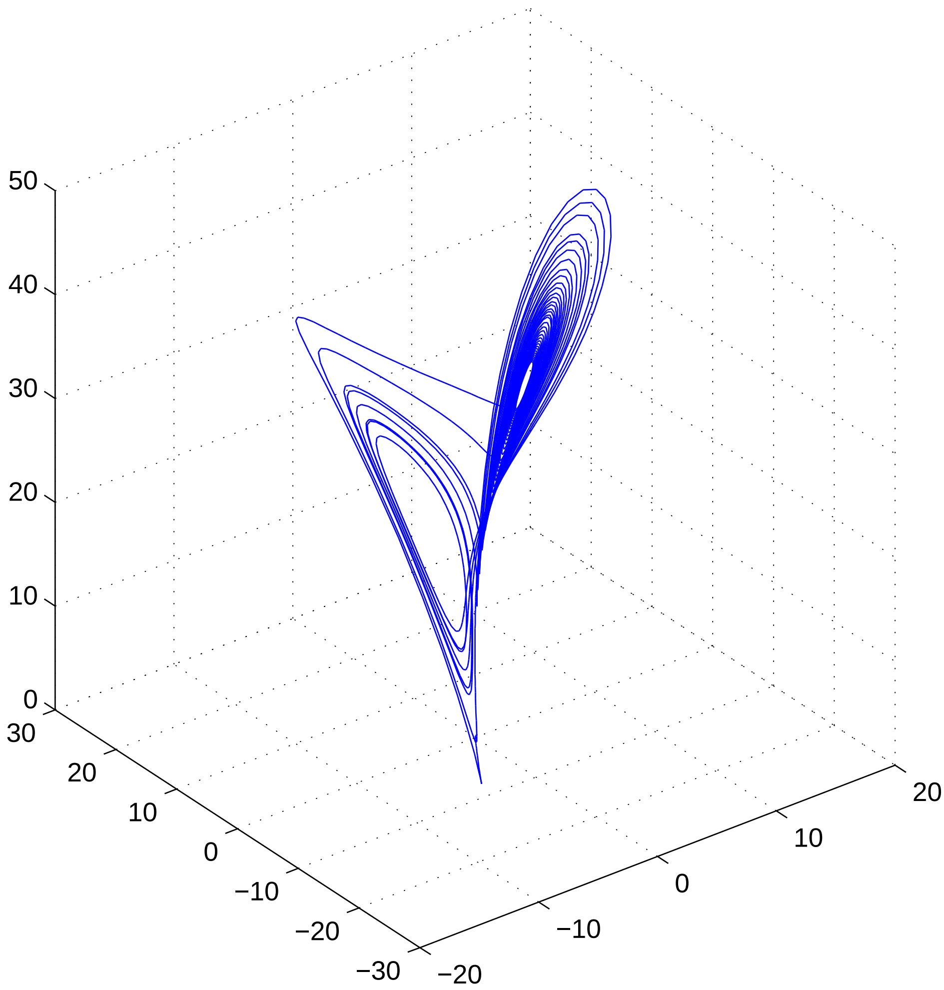
Chaotic orbits are still not possible, for 2D autonomous problems (note, however: 2D nonautonomous problems can be considered 3D autonomous problems). Chaotic orbits are possible in 3D autonomous systems (see Figure 8.9).

3. 3D systems. Now chaotic orbits are possible, for example, in the Lorenz equations ($\sigma = 10, b = 8/3, r = 28.$). Recall that chaotic orbits are bounded and not asymptotically periodic, and sensitive to initial conditions. However, there is here a "chaotic attractor".

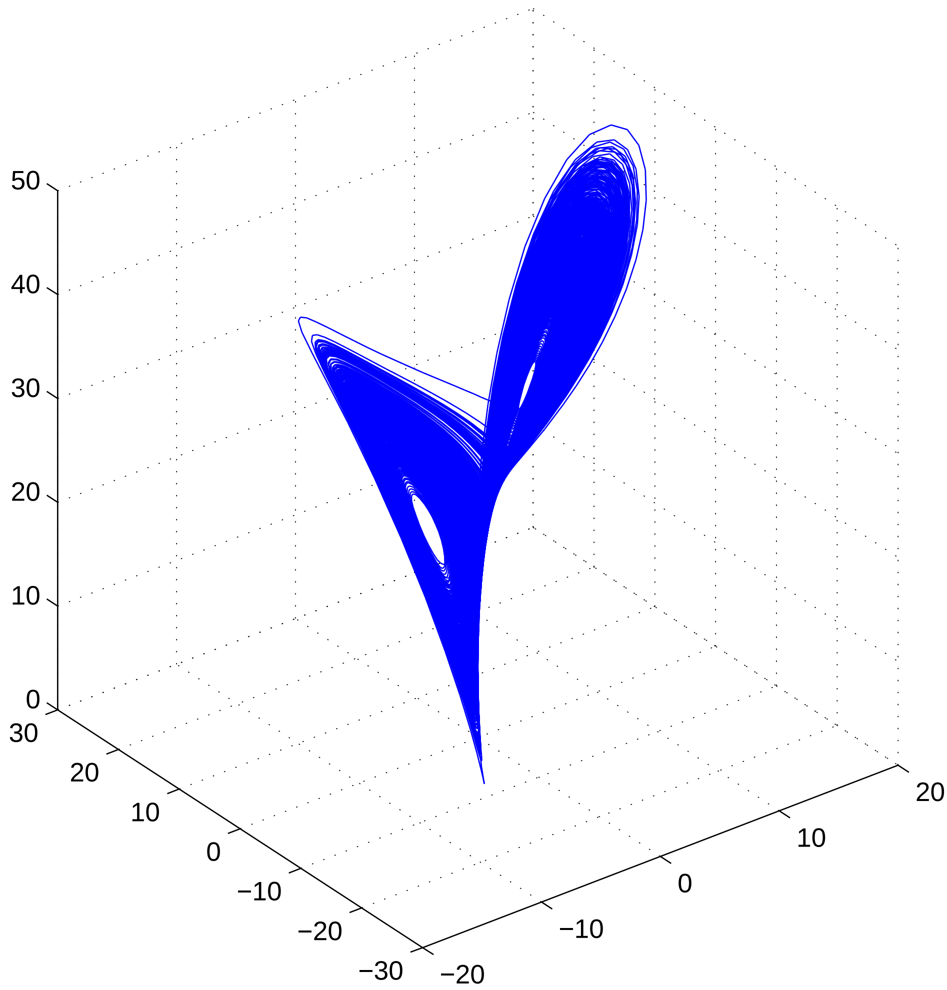
$$\begin{aligned} x'(t) &= -\sigma x + \sigma y \\ y'(t) &= -xz + rx - y \\ z'(t) &= xy - bz \end{aligned}$$



Chaotic orbit $0 < t < 30, x_0 = 0.1, y_0 = -0.2, z_0 = 0.3$



Chaotic orbit $0 < t < 30, x_0 = 0.1001, y_0 = -0.2, z_0 = 0.3$



Chaotic orbit $0 < t < 1000$