Notes on Chapter 1

1 $(v_1, v_2, ...v_m)$ spans V if every element of V can be written as a linear combination $\sum_{k=1}^{m} \alpha_k v_k$.

2 $(v_1, v_2, ...v_m)$ is linearly independent if $\sum_{k=1}^{m} \alpha_k v_k = 0$ implies all $\alpha_k = 0$.

Note: a set is linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

3 $(v_1, v_2, ...v_m)$ is a basis for V if it spans V and is linearly independent.

4 if $(v_1, v_2, ...v_m)$ is linearly independent but doesn't span V, you can add another vector v_{m+1} in V such that $(v_1, v_2, ...v_m, v_{m+1})$ is still linearly independent.

Proof: If these m vectors do not span V, there exists at least one vector that is not a linear combination of the m: call it v_{m+1}. Then assume $\alpha_1 v_1 + ... + \alpha_m v_m + \alpha_{m+1} v_{m+1} = 0$; α_{m+1} must be zero, otherwise v_{m+1} could be written as a linear combination of the other m. Thus $\alpha_1 v_1 + ... + \alpha_m v_m = 0$, which implies that $\alpha_1, ..., \alpha_m$ are 0 also.

5 if $(v_1, v_2, ...v_m)$ spans V but is not linearly independent, you can remove a vector v_l such that $(v_1, ...v_{l-1}, v_{l+1}, ...v_m)$ still spans V.

Proof: One of the m vectors can be written as a linear combination of the others: call it v_l, and remove it. Then any vector that could be written as a linear combination $\alpha_1 v_1 + ... + \alpha_l v_l + ... + \alpha_m v_m$ can still be written as a linear combination of the remaining vectors, because the v_l in this expression can be replaced by its expansion in terms of the remaining vectors.

6 all bases for a (finite dimensional) vector space V have the same number of vectors, dim(V) (proven in Theorem 1.5.1)

7 for a set of vectors $(v_1, v_2, ...v_m)$:

m	independent	spans V	basis for V
$<$ dim(V)	maybe	no	no
= dim(V)	maybe	maybe	maybe
$>$ dim(V)	no	maybe	no

8 when m=dim(V), any one property implies the other two.

a. if m $<$ dim(V), the set cannot span V because: either (a) the set is independent, in which case you have found a basis with fewer than dim(V) elements or (b) the set is dependent, in which case you can remove vectors until you have a basis, with even fewer elements.

b. if m $>$ dim(V), the set cannot be independent because: either (a) the set spans V, in which case you have found a basis with more than dim(V) elements or (b) the set does not span V, in which case you can add vectors until you have a basis, with even more elements.

c. if m = dim(V), and the set is independent, it must span V, because otherwise you could add vectors until it does span V, and have found a basis with more than dim(V) elements.

d. if m = dim(V), and the set spans V, it must be independent, because otherwise you could remove vectors until it is independent, and have found a basis with fewer than dim(V) elements.

9 all norms in R^n are equivalent: $\|z\|_p = \|u\|_p \|z\|_2$ where $u = z/\|z\|_2$ so $\|u\|_2 = 1$. Let m and M be the minimum and maximum of the p-norm on the unit ball (set of vectors with $\|u\|_2 = 1$). This minimum and maximum must exist, because (see Lemma 1.6.2) every norm is continuous and a continuous function on a bounded, closed set has a maximum and minimum. Then $m\|z\|_2 \leq \|z\|_p \leq M\|z\|_2$. (Also, $m > 0$, because $\|u\|_p > 0$ for all nonzero u and a continuous function on a closed and bounded set attains its maximum and minimum.)