Notes on Chapter 2

1. Row rank is dimension of space spanned by rows, column rank is dimension of space spanned by columns of M by N matrix A.

2. Elementary row operations do not change row rank or column rank of a matrix (Corollary 2.3.1 (5)).

3. Row rank and column rank of matrix in RREF are both equal to number of nonzero rows, eg, the following matrix has row and column rank = 3. Thus row rank = column rank = "rank" of original matrix. Note that rank of matrix, k, satisfies $k \leq min(M, N)$.

$$\begin{bmatrix} 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Dimension of range of A = rank(A) = k. Obvious for matrix (EA) in RREF, because EAx=Eb has solutions when $Eb = (c_1, ..., c_k, 0, ...0)$, where $c_1, ..., c_k$ are arbitrary. Since Ax=b if and only if EAx=Eb, dimension of range of A is same as dimension of range of EA.

5. Dimension of null space of A = N-k. Obvious once matrix is in RREF, because you can solve EAx=0 for k "pivot" variables in terms of N-k "non-pivot" variables, which may have arbitrary values. Since Ax=0 if and only if EAx=0, null space of A is same as null space of EA.

6. If $M < N$ (fewer equations than unknowns), null space must have positive dimension, because $k \leq min(M, N) = M < N$, so $N - k > 0$. Thus if Ax=b has a solution it cannot be unique.

7. If $M > N$ (more equations than unknowns), range of A cannot be all of R^M, because $k \leq min(M, N) = N < M$. Thus Ax=b will "usually" have no solution.

8. If M=N (same number of equations as unknowns, usually the case in applications) either:

a. k=N (A is "nonsingular"), so range is all of R^N, and N-k=0 so null space consists of only the zero vector. Thus Ax=b has a solution for all b, and it is unique, because if Ay=b also, A(y-x)=0, and thus y-x=0, or y=x.

b. $k < N$ (A is "singular") so range is not all of R^N and null space has positive dimension. Thus Ax=b usually has no solution, and if it does, it cannot be unique.

9. To summarize:

	Ax=b has solutions for all b (dim(range) = M) (map is "onto" R^M)	solutions unique if exist (dim(null space) = 0) (map is "1-to-1")
$M < N$	maybe	no
$M = N$, A singular	no	no
$M = N$, A nonsingular	yes	yes
$M > N$	no	maybe

10. Now we can prove that all bases in R^M must have exactly M vectors (cf. Theorem 1.5.1). If $(v_1, v_2, ..., v_N)$ form a basis, every vector v in R^M has a unique expansion $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_N v_N$. This means $A\alpha = v$ has a unique solution for any v in R^M, where A is an M by N matrix whose columns are $v_1, v_2, ..., v_N$. But we have just shown above that this can happen only if M=N.