Notes on Chapter 7

1. When we solve a linear system by reducing it to upper triangular form, let us keep track of the multipliers we use to do the reduction. Consider the augmented matrix $(A|b)$, corresponding to the linear system $Ax = b$:

$$\begin{bmatrix} 2 & 3 & -1 & | & 5 \\ 4 & 4 & -3 & | & 3 \\ 2 & -3 & 1 & | & -1 \end{bmatrix}$$

First take -2 times row 1 and add to row 2, but save the negative of the multiplier (ie, +2) in the position that was zeroed. The () reminds us that this position is really 0.

$$\begin{bmatrix} 2 & 3 & -1 & | & 5 \\ (2) & -2 & -1 & | & -7 \\ 2 & -3 & 1 & | & -1 \end{bmatrix}$$

Now take -1 times row 1 and add to row 3, and replace the zeroed element $A_{3,1}$ by the negative of the multiplier used to zero it:

$$\begin{bmatrix} 2 & 3 & -1 & | & 5 \\ (2) & -2 & -1 & | & -7 \\ (1) & -6 & 2 & | & -6 \end{bmatrix}$$

Finally, take -3 times row 2 and add to row 3, and save the negative of the multiplier used:

$$\begin{bmatrix} 2 & 3 & -1 & | & 5 \\ (2) & -2 & -1 & | & -7 \\ (1) & (3) & 5 & | & 15 \end{bmatrix}$$

Finish the back substitution, to find the solution $x = (1, 2, 3)$.

2. Each row operation used to reduce A to upper triangular form U, was equivalent to multiplying both sides of $Ax = b$ by an elementary matrix $M_{i,j}$. In this problem:

$$M_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$M_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Thus we reduced $Ax = b$ to $M_{3,2}M_{3,1}M_{2,1}Ax = M_{3,2}M_{3,1}M_{2,1}b$, where $M_{3,2}M_{3,1}M_{2,1}A = U$, the final upper triangular matrix. This means that $A = (M_{2,1}^{-1}M_{3,1}^{-1}M_{3,2}^{-1})U = LU$, where L is lower triangular, because each $M_{i,j}^{-1}$ is lower triangular ($M_{i,j}^{-1}$ is the same as $M_{i,j}$ except the sign of the off-diagonal term is reversed). In fact, we can compute that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Now we see that by saving the negatives of the multipliers above, we have essentially saved L as well as U (the final upper triangular matrix):

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

3. Now if we need to solve another linear system $Ax = c$ with the same matrix A but different right hand side vector, we can do it with much less computer time (for large problems), by solving $LUx = c$ in two stages, first $Ly = c$, then $Ux = y$; both are triangular systems so they are easily solved. For example, if $c = (-1, 3, -1)$ we can solve $Ax = c$ first by solving $Ly = c$ using forward substitution (solve first equation for y_1, etc.):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

which gives $y = (-1, 5, -15)$. Now use back substitution to solve $Ux = y$:

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -15 \end{bmatrix}$$

which gives $x = (-0.5, -1, -3)$.

4. Now let us solve a linear least squares example $min\|Ax-b\|_2$. We attempt to find a solution of $Ax=b$, where:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

We proceed to reduce this to upper triangular form, but now using orthogonal Givens rotations rather than elementary matrices. First, to zero the element $A_{2,1}$ we multiply both sides by

$$Q_{2,1} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $c = \frac{1}{\sqrt{2}}$, $s = \frac{1}{\sqrt{2}}$, and this gives:

$$\begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} \\ 1/\sqrt{2} \\ 2 \end{bmatrix}$$

Next, to zero $A_{3,1}$ we multiply both sides by

$$Q_{3,1} = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

where $c = \frac{\sqrt{2}}{\sqrt{3}}$, $s = \frac{1}{\sqrt{3}}$ and this gives:

$$\begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{bmatrix}$$

Finally, to zero $A_{3,2}$ we multiply by

$$Q_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

where $c = \frac{1}{2}$, $s = \frac{\sqrt{3}}{2}$ and this gives:

$$\begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{3} \\ 1/\sqrt{2} \\ -1/\sqrt{6} \end{bmatrix}$$

The least squares solution now is obvious, we simply ignore the last equation (which we can't control anyway) and solve the first two exactly by back substitution, and get $x = (\frac{2}{3}, \frac{1}{2})$.

5. Notice we solved the least squares problem by multiplying both sides of $Ax = b$ by orthogonal matrices, to reduce to an upper triangular form where the least squares solution was obvious (and remember that multiplying by orthogonal matrices does not change the least squares solution). That is, we chose orthogonal matrices so that $Q_{3,2}Q_{3,1}Q_{2,1}Ax = Q_{3,2}Q_{3,1}Q_{2,1}b$, was an upper triangular system, $Rx = c$. Then if we define $Q^T = Q_{3,2}Q_{3,1}Q_{2,1}$, we have $Q^T A = R$, or $A = QR$; this is called the QR decomposition of A. It is used for the same purpose as the LU decomposition: suppose we now want to solve another least squares problem min $\|Ax - d\|_2$, with the same A but different d. Then the system $Ax = d$ which we want to approximately solve can be written $QRx = d$, or $Rx = Q^T d$ and to solve this we simply multiply d by Q^T, then ignore the rows of the triangular system $Rx = Q^T d$ that have all zeros on the left hand side, and solve the remaining ones using back substitution.