Oct 31 Homework

Some of the problems below don't really fit into chapter 11, but that's ok, this is a course on mathematical problem solving, and these are all mathematical problems!

1. A fascinating result from probability theory is that if you start with ANY probability distribution, take N random samples and average them, and repeat this many times, if N is large, these averages will distribute themselves around the mean μ with a "normal" distribution. More precisely, the variable $z = \frac{x-\mu}{\sigma/\sqrt{N}}$, where σ is the standard deviation of the original distribution, will have a normal distribution $n(z) = e^{-z^2/2}/\sqrt{2\pi}$. In other words, no matter what shape the original distribution has, the distribution of averages will be nearly a normal curve, with the same average, but a standard deviation $\sqrt{N}$ times smaller. (Hence, a poll of 400 voters has about half the margin of error that a poll of 100 voters has.) Show that $I \equiv \int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}$, so that $n(z)$ is really a probability distribution. (Hint: write I^2 as a double integral, over the entire plane, and convert to polar coordinates.) It seems quite remarkable that the two most famous irrational numbers in mathematics, e and π, both show up in the "central limit theorem" of probability, which seems far removed from the natural habitat of e (calculus) or of π (geometry).

2. Schumer problem 11.6 (a-c).

3. As mentioned in the text, Colin Percival computed π to 10^{15} binary digits, in 2000.

    a. Calculate how many terms N of the series for arctan(1) (11.6) would be required to get that much accuracy. Express your answer in the form 10^n. (Hint: For a monotone alternating series, the error after N terms is always less than the next term, in absolute value.)

    b. Calculate how many terms N of the series for arctan(1/3) would be required to get that much accuracy. Express your answer in the form 10^n again. Since the expansion of arctan(1/3) is the slower convering series, this shows that the work to calculate π using the Hutton formula from problem 11.6c is dramatically less than for the straightforward arctan series (11.6).

4. A set of numbers is said to be "countable" if you can list them all in a (possibly infinite) sequence. For example, the integers are countable because you can list them all as follows: 0,-1,1,-2,2,-3,3,... The set of all real numbers between 0 and 1 is not countable, because if you give me any list of these numbers, I can find one which is nowhere in your list: write out the decimal representation for each number (which will be infinitely long in many cases), and I can construct a number which differs in the first decimal place from your first number, differs in the second place from your second number, and so on, and thus is different from every number in your list.

   a. If you try to use the above proof to show that the set of all rational numbers between 0 and 1 is not countable, what goes wrong?

   b. In fact, show that the set of rational numbers between 0 and 1 is countable, by indicating how one can list them. Thus although there are an infinite number of rational numbers between 0 and 1, and an infinite number of irrational numbers, in a real sense there are many more irrational numbers.

   c. Is the set of points in the xy plane of the form (p,q), where p and q are rational numbers between 0 and 1, countable? Justify your answer.

5. (Extra credit) Consider the function $u(x, y) = x+y$, and notice that $\frac{\partial u}{\partial x} = 1$. Now make the change of variables:

$$\begin{aligned} x &= p \\ y &= p + q \end{aligned}$$

   and note that $u(p, q) = x + y = p + (p + q) = 2p + q$, so that $\frac{\partial u}{\partial p} = 2$. But since x and p are always equal, how can $\frac{\partial u}{\partial p}$ be different from $\frac{\partial u}{\partial x}$??

6. (Extra credit!) In our text, the claim is made that the BBP formula for π, shown on page 113, allows one to "jump" to the n^{th} hexadecimal digit, since the series is of the form $\sum_{n=0}^{\infty} \frac{a_n}{16^n}$. If the a_n were integers it would indeed be easy to jump to a given hex digit, but most of the coefficients a_n do not even have terminating hexadecimal representations, so how is it easier to extract the n^{th} hex digit of π using the BBP formula, than using any of the other formulas in this chapter?

Since writing the above, a student has pointed me to an explanation of the use of the BBP formula, at:

http://en.wikipedia.org/wiki/Bailey-Borwein-Plouffe_formula

Certainly, saying this formula allows one to "jump" to the n^{th} hexadecimal digit is an exaggeration. Basically, if you want, say the 1000000^{th} hex digit of π, you multiply both sides of the BBP formula on page 113 by 16^{999999}, then the 1000000^{th} hex digit becomes the first digit after the decimal point, so to find this digit you simply throw away the integer part of each of the first 1000000 terms in the new series (the remaining terms will have no integer parts). But you still have to compute the contributions of the first million terms to the fractional part, and an unpredictable small number of terms beyond that. Normally one or two beyond the millionth term will be sufficient, but if you are very unlucky and the sum of the first million terms has fractional part $0.8FFFFFFFFFFF_{16}$ then you may need to compute another 10 terms or so before you can be sure whether the first hex digit is an 8 or 9. Further, even though you have thrown away the integer parts of each term in the series, the sum of the fractional parts will probably not be between 0 and 1, so at the end, you still have to add or subtract an integer to make it between 0 and 1.

But the above procedure could be used to compute the 1000000^{th} digit of π using any rapidly convergence series, so what is special about the BBP formula? After the series has been multiplied by 16^{999999}, the contribution of the k^{th} term (let's just look at the first of the 4 parts of the BBP formula) to the fractional part is computed as $(4 * 16^{999999-k})_{mod(8k+1)}/(8k+1)$, but it isn't obvious, and isn't explained in the "wikipedia" reference, how one can compute this without resorting to very long (4000000 bit) integer data types, as normally would be required, or why the fractional parts of these terms are easier to compute than the fractional parts of the terms of say, series (11.7), after multiplication by 16^{999999}. Extra credit for you if you can now explain how the BBP formula makes computing the millionth digit of π easier than formulas such as (11.7). (Hint: this is related to a problem in your Sept 26 homework.)