Hyperboloidal Model
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Theory

A quadric surface is a surface defined by an equation of the second degree in three variables. The general equation for such a surface is given by

\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + K = 0, \]

where \( A, B, C, D, E, F, G, H, I, \) and \( K \) are Real constants.
CONICIODS

A *quadric surface* is a called a conicoid because any plane section of a quadric surface produces a conic section (parabola, ellipse, and hyperbola) or a limiting case of one (lines or a point).

- **Circle or Ellipse**
- **Hyperbola**
- **Parabola**
The Two Hyperboloids

Hyperboloid of one sheet

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]  

Hyperboloid of two sheet

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

There are two separate hyperboloids as shown above. Both cases are shown in the figures above, where \( a, b, \) and \( c \) are Real constants.
The hyperboloid of 1 sheet, as shown above in figure 3, sometimes described as an elliptical hyperboloid, has ellipses for plane sections parallel to the $xy$-plane.

Plane sections parallel to the $z$-axis are hyperbolas.
Hyperboloid Traces

The three equations for the traces in the principal planes of the hyperboloid of one sheet are shown above.

\begin{align*}
  xz\text{-trace: } (2) & \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\
  xy\text{-trace: } (3) & \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \\
  yz\text{-trace: } (4) & \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
\end{align*}

Hyperboloid of one sheet

\begin{align*}
  (1) & \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
\end{align*}

The three equations for the traces in the principal planes of the hyperboloid of one sheet are shown above.
When $a = b$ in equation (1), the elliptical hyperboloid of one sheet becomes the hyperbola of revolution and the traces parallel to the $xy$-plane become circles. In the model for part C, $a = b = 4$ and $c = 5$. By substituting these values into (1), we get (5), the equation for the surface for the model in part C.

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
$$

$$
\frac{x^2}{16} + \frac{y^2}{16} - \frac{z^2}{100} = 1
$$
Applications of the Hyperboloid

There are simple applications for the hyperboloid in the form of metal or natural fibers.

Another simple application is the use of wooden laths to build circular-form trellises for the garden.

In all of these applications, the hyperboloid is one of revolution and the structure is a ruled surface. The entire structure can be formed with straight line-segment elements and circular cross-sectional parts for support.
Another Application

The nuclear cooling towers are semi-hyperboloids. They approximate the hyperboloid of revolution and help exhaust hot air from the reactor.
The beautiful Gothic cathedral in Barcelona, Spain has large hyperboloid supported by 4-large central pillars, surrounded by 12 smaller hyperboloid structures. The cathedral was started in 1882 and the famous architect Antoni Gaudi (1852-1926) developed its basic design.
Kobe Port Tower is a hyperboloid structure, 108 m high. It is a lattice tower in the port city of Kobe, Japan. Observe the red-steel beams emphasizing that it is a ruled surface.

It has an observation deck which offers spectacular views of the bay and the surrounding area.

The tower was completed in 1963.
The theory of the hyperboloid was developed in part A. It was shown that the hyperboloid of revolution is a special case. You will use equation (5) from part A as the basis for constructing your model.

\[
\frac{x^2}{16} + \frac{y^2}{16} - \frac{z^2}{100} = 1
\]
Basic Tools You Need for the Hyperboloid Model

Compass

Protractor

Ruler (with centimeter scale)

Scissors
Basic Materials You Need for the Hyperboloid Model

Paper for Diagram:
any plane paper or white posterboard, about 14 in x 19 in
(40 cm x 50 cm)

Posterboard for Parts:
(4-ply or 6-ply)

1 sheets poster board in one color, color on both sides, about 22 in x 28 in
(55 cm x 70 cm)

1 sheets poster board in contrasting color, color on both sides, about 22 in x 28 in
(55 cm x 70 cm)
You will construct a template similar to the one shown in *figure 4*. The detailed construction is shown on the pages which follow.

From the diagram of *figure 4*, you will take measurements to construct 14 circular disks which interlock with slots to form the model.
The 14 slotted disks shown represent the parts of your model. The parts are constructed using the basic diagram of figure 5.
Draw two perpendicular lines, centered on your paper. Use a compass or a protractor for accuracy.
Mark 6 equal spaces of 6 cm along the horizontal axis. Then, mark 6 equal spaces of 4 cm on the vertical axis, as shown in figure 7.
ADD PARALLEL LINES

Pass the 6 parallel lines through the marked spaces as shown in figure 8.
Pass the 6 more parallel lines through the marked spaces as shown in figure 9.
ADD MORE PARALLEL LINES

Pass the 6 more parallel lines through the marked spaces as shown in figure 9.
ADD MORE PARALLEL LINES

Use a ruler or compass to find the midpoint of AB at M and the midpoint of BC at N as shown in figure 10.
Pass a line through the points O and M. Then, pass a line through the points O and N as shown in figure 11.
After you complete all of the parallel lines in your diagram, remove the letters A, B, C, M, N and leave O, as shown in figure 12. This is a necessary step so that the same letters can be used in the steps to follow, but for different points.
Label the vertical axis with variable $z$ and the horizontal axis with variable $x$. You will use a centimeter ruler to plot data for a hyperbola curve, shown on the next page, page 27, in *figure 14.*
You will calculate data and make a hyperbola graph on the grid you made in figure 13. When you finish you will have the diagram similar to the one in figure 14.
EQUATION FOR THE MODEL

\[ \frac{x^2}{16} + \frac{y^2}{16} - \frac{z^2}{100} = 1 \]  \hspace{1cm} (5)

When \( y = 0, \)

\[ \frac{x^2}{16} - \frac{z^2}{100} = 1 \]  \hspace{1cm} (1)

Equation (5) represents is the mathematical equation representing the surface of the model you build. It was developed on page 13 of part A, where the theory of hyperboloids was developed.

When \( y = 0 \) in (5), it simplifies to the \( xz \)-trace, the equation of the graphy you will need for a template. The equation is designated as equation (1) above.
SOLVE THE EQUATION

\[
\frac{x^2}{16} - \frac{z^2}{100} = 1
\]

Solve equation (1) above for \( z \). The solution is shown on the next page.
Equation (1) is solved in the steps above. You use equation (9) to compute the data needed for the hyperbola template shown in figure 14, on page 27.
(9) \[ z = \pm \frac{5}{2} \sqrt{x^2 - 16} \]

**Example Calculation:**

If \( x = \pm 5 \),

(10) \[ z = \pm \frac{5}{2} \sqrt{(\pm 5)^2 - 16} \]

\[ z = \pm \frac{5}{2} \sqrt{25 - 16} = \pm \frac{5}{2} \sqrt{9} = \pm \frac{5}{2} \cdot 3 = \pm 7.5 \]

\((x,z) = (\pm 5, \pm 7.5)\) or \((x,z) \in \{(5, 7.5),(5, -7.5),(-5, 7.5),(-5, -7.5)\}\)

Equation (9) is the equation for hyperbola template shown in *figure 14*, on page 27. You need to compute values for \((x,z)\) and plot the graph on your grid.
Complete the Table 1. The values for \( x = \pm 5 \) were computed in the sample on page 31.

Use a calculator to approximate any of the square roots to the nearest tenth.

Check your answers on page 33.

<table>
<thead>
<tr>
<th>( x ) (cm)</th>
<th>( z ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>± 4</td>
<td></td>
</tr>
<tr>
<td>± 5</td>
<td>± 7.5</td>
</tr>
<tr>
<td>± 6</td>
<td></td>
</tr>
<tr>
<td>± 7</td>
<td></td>
</tr>
<tr>
<td>± 8</td>
<td></td>
</tr>
</tbody>
</table>

**Table 1**
### CHECK YOUR DATA

Table 1 is shown completed for you to check your answers.

\[ z = \pm \frac{5}{2} \sqrt{x^2 - 16} \]

<table>
<thead>
<tr>
<th>( x ) (cm)</th>
<th>( z ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>± 4</td>
<td>0</td>
</tr>
<tr>
<td>± 5</td>
<td>± 7.5</td>
</tr>
<tr>
<td>± 6</td>
<td>± 11.2</td>
</tr>
<tr>
<td>± 7</td>
<td>± 14.4</td>
</tr>
<tr>
<td>± 8</td>
<td>± 17.3</td>
</tr>
</tbody>
</table>

**Table 1**
Hyperbola Graph

\[ z = \pm \frac{5}{2} \sqrt{x^2 - 16} \]

Plot the data you calculated in Table I on your basic grid.

Use a centimeter ruler to locate the points on the graph.

The \((x,z) = (6,11.2)\) is shown measured from the axes.
**Label Points**

In *figure 16*, the segments AB, CD, EF, and GH have additional points labeled from *figure 15*.

Label the intersecting points on your diagram to match those shown: I, O, J on segment AB; K, L, M on segment CD; N, P, Q on segment EF; and R and S on segment GH.
**MEASURE**

Use a centimeter ruler to measure the lengths of the segments: $AB$, $CD$, $EF$, and $GH$. Write the lengths of the segments in Table 2.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td></td>
</tr>
<tr>
<td>$CD$</td>
<td></td>
</tr>
<tr>
<td>$EF$</td>
<td></td>
</tr>
<tr>
<td>$GH$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 2**

![Figure 17](image_url)
Divide the Lengths

From Table 2, take half of the lengths of each segment you measured and write the number in the spaces provided in Table 3.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Length (cm)</th>
<th>$\frac{1}{2}$ Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$CD$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$EF$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$GH$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

figure 18
Use the radius for AB from Table 3 to construct the two circular disks shown in figure 19. Draw a diameter on the disk and label the end points A and B. Then use your basic grid of figure 17 to locate the points I, O, and J by measuring the spaces from the grid and marking them on the diameter of AB.
Making the Slots for Parts 1 and 2

Use a protractor or compass to construct perpendicular lines at points I, O, and J. Then, cut very thin slots using a scissors. You need to cut twice to make the small space on each slot, as shown in figure 20.
Constructing Parts 3, 4, 5, and 6

Use the radius for CD from Table 3 to construct the four circular disks shown in figure 21. Draw a diameter on the disk and label the end points C and D. Then use your basic grid of figure 17 to locate the points K, L, and M by measuring the spaces from the grid and marking them on the diameter of CD.
Making the Slots for Parts 3, 4, 5, and 6

Use a protractor or compass to construct perpendicular lines at points K, I, and M. Then, cut very thin slots using a scissors. You need to cut twice to make the small space on each slot, as shown in figure 21.
Constructing Parts 7, 8, 9, and 10

Use the radius for EF from Table 3 to construct the four circular disks shown in figure 22. Draw a diameter on the disk and label the end points E and F. Then use your basic grid of figure 17 to locate the points N, P, and Q by measuring the spaces from the grid and marking them on the diameter of EF.
Making the Slots for Parts 7, 8, 9, and 10

Use a protractor or compass to construct perpendicular lines at points N, P, and Q. Then, cut very thin slots using a scissors. You need to cut twice to make the small space on each slot, as shown in figure 23.
Use the radius for GH from Table 3 to construct the four circular disks shown in figure 24. Draw a diameter on the disk and label the end points E and F. Then use your basic grid of figure 17 to locate the points R and S by measuring the spaces from the grid and marking them on the diameter of GH.
Making the Slots for Parts 11, 12, 13, and 14

Use a protractor or compass to construct perpendicular lines at points R and S. Then, cut very thin slots using a scissors. You need to cut twice to make the small space on each slot, as shown in figure 25.
Figure 17 is very important in assembling the 14 slotted disks of the model. The disks slide together by positioning the slots as shown in figure 26. Start from the center with the two smallest disks, part 1 and part 2. The orange disks are parallel to each other and the blue disks are parallel to each other.
When you position all of the disks in their slotted positions, they match the diagram of *figure 17* and you have completed the model. This is illustrated by *figure 27*.