1.2 Functions and Models

To mathematically model a situation means to represent it in mathematical terms. The particular representation used is called a mathematical model of the situation. Mathematical models do not always represent a situation perfectly or completely. Some represent a situation only approximately, whereas others represent only some aspects of the situation.

Analytic models are obtained by analyzing the situation being modeled. Curve-fitting models are obtained by finding mathematical formulas that approximate observed data. Continuous models are defined by functions whose domains are intervals of the real line. A discrete model as its domain is a discrete set. Discrete models are used extensively in probability and statistics.

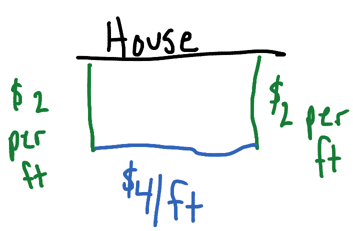
This is a major focus of our text, so get used to applications and word problems. Since the main student population for this class is headed in to a business related field, most of our examples will be business related as well.

Example: The amount of free space left on your hard drive is 50GB and is decreasing by 5 GB per month. Find a mathematical model for this situation.

Start at 50 subtract 5 each month (m)

$$F = 50 - 5m$$

Example: My square orchid garden abuts my house so that the house itself forms the northern boundary. The fencing for the southern boundary costs \$4 per foot, and the fencing for the east and west sides cost \$2 per foot. Find the total cost of the function as a function of the length of a side *x*.



$$C(x) = 4x + 2x + 2x$$

which becomes

$$C(x) = 8x$$

Definition – A cost function specifies the cost *C* as a function of the number of items *x*. Thus, $C(x)$ is the cost of *x* items, and has the form: Cost = variable cost + fixed cost, where the variable cost is a function of *x* and the fixed cost is constant. A cost function of the form $C(x) = mx + b$ is called a linear cost function; the variable cost is *mx* and the fixed cost is *b*. The slope *m*, the marginal cost, measures the incremental cost per item.

Definition – The revenue resulting from one or more business transactions is the total payment received, sometimes called the gross proceeds. If $R(x)$ is the revenue from selling *x* items at a price of *m* each, then *R* is the linear function $R(x) = mx$ and the selling price *m* can also be called the marginal revenue. Notice that there is no fixed revenue; you are not guaranteed to make money.

Definition – The profit, on the other hand, is the net proceeds, or what remains of the revenue when costs are subtracted. If the profit depends linearly on the number of items, the slope *m* is called the marginal profit. Profit, revenue, and cost are related by the following formula: Profit = Revenue – Cost, or, $P = R - C$.

Definition – The break-even point is the number of items *x* at which profit is zero, or revenue is equal to cost.

Example: The Metropolitan Company sells its latest product at a unit price of \$5. Variable costs are estimated to be 30% of the total revenue, while fixed costs amount to \$7,000 per month. How many units should the company sell per month in order to break even, assuming that it can sell up to 5,000 units per month at the planned price?

Sells $\Rightarrow$ revenue $R(x) = 5x$ where x is the number of units

Cost $\Rightarrow$ $C(x) = 30\%$ of $R + 7000$

$C(x) = 0.30(5x) + 7000 \longrightarrow C(x) = 1.5x + 7000$

Break-even $R(x) = C(x)$

$$\begin{aligned} 5x &= 1.5x + 7000 \\ -1.5x & \quad -1.5x \\ \hline 3.5x &= 7000 \end{aligned}$$

$x = \dfrac{7000}{3.5}$

$x = 2000$ units to break-even

Definition – A demand equation or demand function expresses demand *q* as a function of the unit price *p* (price per item). A supply equation or supply function expresses supply *q* as a function of the unit price *p*. It is usually the case that demand decreases and supply increases as the unit price increases. Equilibrium occurs when supply is equal to demand.

Example: The demand for your hand-made skateboards, in weekly sales, is $q = -3p + 700$ if the selling price is \$*p*. You are prepared to supply $q = 2p - 500$ per week at the price \$*p*. At what price should you sell your skateboards so that there is neither a shortage nor a surplus?

find \$p

demand is $q = -3p + 700$ supply is $q = 2p - 500$

supply = demand

$$-3p + 700 = 2p - 500$$
$$+3p \quad +500 \qquad +3p \quad +500$$
$$1200 = 5p$$

$$\frac{1200}{5} = p$$

$p = \$240$ per board

Compound Interest Formula – If an amount (present value) *P* is invested for *t* years at an annual rate of *r*, and if the interest is compounded (reinvested) *m* times per year, then the future value *A* is

$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$. A special case is interest compounded once per year: $A(t) = P(1+r)^t$.

Example: Jack invested \$5000 into an account that pays 7.5% per year compounded monthly. How much will he have in 7 years?

$m = 12$ (monthly)

$r = 7.5\% = 0.075$

$$A = 5000\left(1 + \frac{.075}{12}\right)^{(12\cdot 7)}$$

$$A = 5000(1.00625)^{84} = 8438.49598153$$

$A = \$8438.50$ ← round money to 2 decimal places

Example: If Jack invested his \$5000 into an account that pays 8% per year compounded once per year, would he have more or less money in the account, compared to the previous example, after 7 years.

$m = 1$ so use special formula

$$A = 5000(1 + .08)^7$$

$$A = 5000(1.08)^7 = 8569.121343$$

$$A = \$8569.12$$

Jack would have more money in this account than in the previous one.

Algebra of Functions – If f and g are real-valued functions of the real variable x, then we define their sum, difference, product and quotient as follows:

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
(f-g)(x) &= f(x)-g(x) \\
(fg)(x) &= f(x)g(x) \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)},\ \ g(x)\neq 0
\end{aligned}$$

Also, if f is as above and c is a constant (real number), then we define the associated constant multiple of f by $cf(x)$.