Chapter Two: Nonlinear Functions and Models

2.1 Quadratic Functions and Models.

Definition – A quadratic function of the variable *x* is a function that can be written in the form $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$ where *a, b,* and *c* are fixed numbers (with $a \neq 0$).

Features of the Graph:

1. The graph of a quadratic function is a parabola.

    a) If $a > 0$ the parabola opens up.

    b) If $a < 0$ the parabola opens down.

2. The vertex is the turning point of the parabola (either the maximum or minimum point). The *x*-coordinate of the vertex is given by $\dfrac{-b}{2a}$. The corresponding *y*-coordinate can be found by $f\left(\dfrac{-b}{2a}\right)$.

3. The *x*-intercepts (if any) occur when $f(x) = 0$. We can generally find these using the quadratic formula. {The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.}

4. The *y*-intercept occurs when $x = 0$; that is, at the point $(0, c)$.

5. The parabola is symmetric with respect to the vertical line through the vertex, which is the line $x = \dfrac{-b}{2a}$.

Examples: Sketch the graph finding all intercepts, the vertex, and the line of symmetry.

1. $f(x) = x^2 + 3x + 2$

$a = 1, b = 3, c = 2$

Vertex $x = -\dfrac{b}{2a} = -\dfrac{3}{2(1)} = -\dfrac{3}{2}$

$y = f(-\tfrac{3}{2}) = -\dfrac{1}{4}$

$\longrightarrow (-\tfrac{3}{2}, -\tfrac{1}{4})$

$\longrightarrow$ y-intercept is $(0,2)$

$\longrightarrow$ Line of Symmetry $x = -\dfrac{3}{2}$

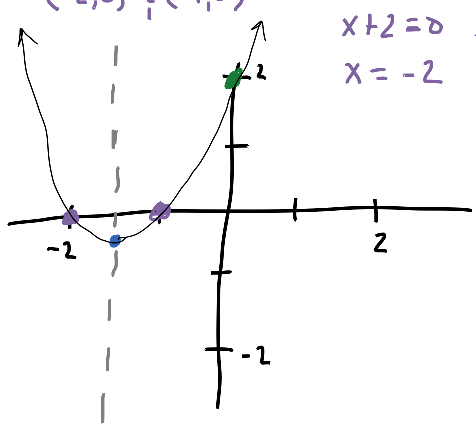
X-intercept(s) $(-2,0)$ & $(-1,0)$

$0 = x^2 + 3x + 2$

$0 = (x+2)(x+1)$

$x+2=0 \quad x+1=0$

$x=-2 \quad x=-1$



2. $f(x) = -x^2 + 4x - 4$ $a=-1, b=4, c=-4$

vertex: $x = \frac{-b}{2a} = \frac{-(4)}{2(-1)} = \frac{-4}{-2} = 2$

$y = f(2) = 0$

$\longrightarrow (2,0)$

$\longrightarrow$ y-intercept $(0,-4)$

$\longrightarrow$ line of symmetry $x=2$

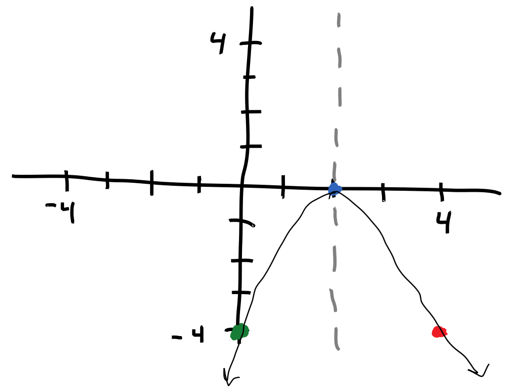
Symmetric point $(4,-4)$

x-intercepts $0 = -x^2 + 4x - 4$

$(2,0)$
only one

$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(-1)(-4)}}{2(-1)}$

$x = \frac{-4}{-2} = 2$



3. $f(x) = x^2 - 10x - 60$ $a=1, b=-10, c=-60$

vertex $x = \frac{-b}{2a} = \frac{-(-10)}{2(1)} = \frac{10}{2} = 5$

$y = f(5) = -85$

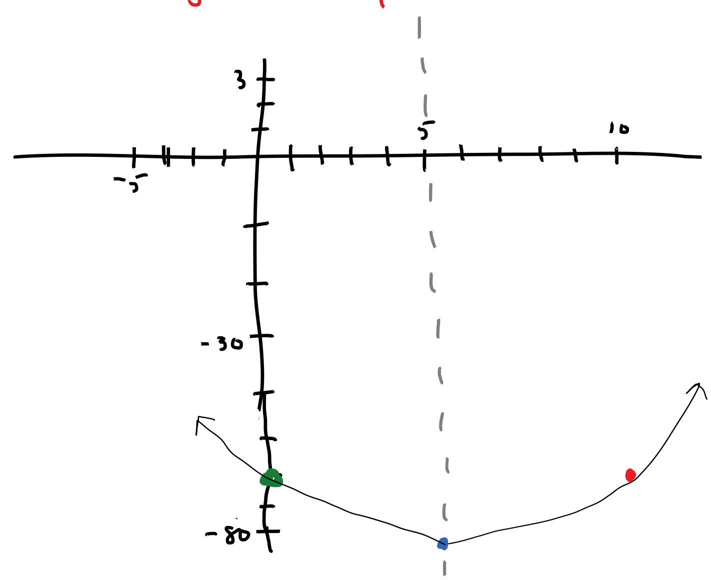
$\longrightarrow (5,-85)$

$\longrightarrow$ y-int $(0,-60)$

$\longrightarrow$ Line of symmetry $x=5$

x-int: $0 = x^2 - 10x - 60$
too hard!

Symmetric point $(10,-60)$



4. You try it: $f(x) = -x^2 - x$

Examples: For each demand equation, express the total revenue *R* as a function of the price *p* per item. Sketch the graph of the resulting function, and determine the price *p* that maximizes total revenue in each case. $R = pq$

1. $q = -3p + 300$

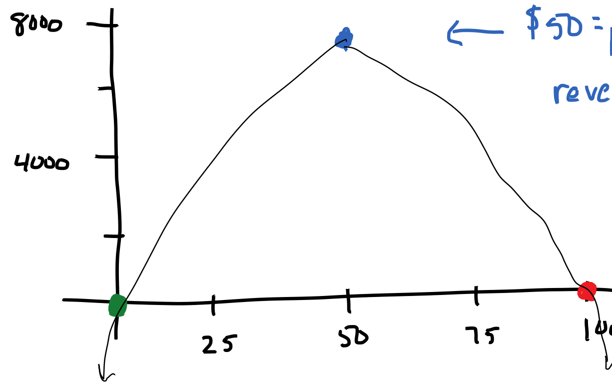
$R(p) = -3p^2 + 300p$

vertex: $p = \frac{-300}{2(-3)} = \frac{-300}{-6} = 50$

$R(50) = 7500$

y-int $(0,0)$

Symmetric point $(100,0)$



← \$50 = p maximizes revenue.

2. $q = -5p + 1200$

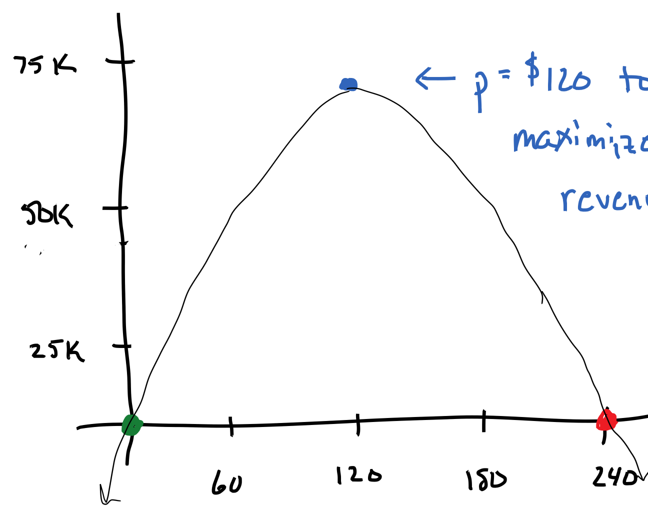
$R(p) = -5p^2 + 1200p$

vertex: $p = \frac{-1200}{2(-5)} = \frac{-1200}{-10} = 120$

$R(120) = 72{,}000$

y-int $(0,0)$

Symmetric point $(240,0)$



← p = \$120 to maximize revenue

Example: Daily oil production by Pemex, Mexico's national oil company, for 2001 – 2009 can be approximated by $P(t) = -0.022t^2 + 0.2t + 2.9$ million barrels/day $(1 \le t \le 9)$ where t is the time in years since the start of 2000. According to the model, in what year was oil production by Pemex greatest? How many barrels per day were produced that year?

find time

vertex

$$\text{vertex: } t = \frac{-b}{2a} = \frac{-(0.2)}{2(-0.022)} = \frac{-0.2}{-0.044} = 4.\overline{54}$$

$$P(4.\overline{54}) = 3.3\overline{54}$$

4.5 years after 2000, so in mid 2004 greatest oil production was 3.35 million barrels per day.

Example: Pack-Em-In Real Estate is building a new housing development. The more houses it builds, the less people will be willing to pay, due to crowding and smaller lot sizes. In fact, if it builds 40 houses in this particular development, it can sell them for \$200,000 each, but if it builds 60 houses, it will only be able to get \$160,000 each. Obtain a linear demand equation and hence determine how many houses Pack-Em-In should build to get the largest revenue. What is the largest possible revenue?

$q(p) = mp + b$

price is input

40 houses → 200,000 $(200000, 40)$

60 houses → 160,000 $(160000, 60)$

$$m = \frac{60-40}{160000-200000} = \frac{20}{-40000} = -\frac{1}{2000}$$

If $m = -\frac{1}{2000}$ use any point to find b:

$$40 = -\frac{1}{2000}(200{,}000) + b$$

$$40 = -100 + b$$

$$+100 \qquad +100$$

$$140 = b$$

demand is $q(p) = -\frac{1}{2000}p + 140$

$$\text{Revenue} = p \cdot q = p\left(-\frac{1}{2000}p + 140\right)$$

$$R(p) = -\frac{1}{2000}p^2 + 140p$$

largest = vertex

$$p = \frac{-b}{2a} = \frac{-140}{2\left(-\frac{1}{2000}\right)} = \frac{-140}{-\frac{1}{1000}} = 140{,}000$$

$$q(140{,}000) = -\frac{1}{2000}(140000) + 140 = 70 \text{ houses}$$

$$R(140{,}000) = \$9{,}800{,}000$$