7.4 Permutations and Combinations

Definition: A permutation of *n* items is an ordered list of those items. The number of possible permutations of *n* items is given by *n* factorial, which is $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$. That is, *n*! is *n* multiplied by every number lower than it. Note that the factorial is only defined for positive integers with 0! = 1.

Definition: A permutation of *n* items taken *r* at a time is an ordered list of *r* items chosen from a set of *n* items. The number of permutations of *n* item taken *r* at a time is given by $P(n,r) = \dfrac{n!}{(n-r)!}$.

Example: The number of permutations of 6 items taken three at a time is 6 x 5 x 4 = 120.

Fact: Permutations require order. Think of placing in a race: order matters. When 6 people start the race, all 6 have the possibility of coming in first place. After that first person crosses the line, only 5 people remain that could come in second place. After the first two cross the line, only 4 people remain that could cross the line in third place.

When order does not matter we use combinations.

Definition: The number of combinations of *n* items taken *r* at a time is given by $C(n,r) = \dfrac{n!}{r!(n-r)!}$.

The *r!* in the denominator eliminates any repeated combinations. That is, for permutations {a, b, c} is different than {b, a, c} or even {c, b, a}. However, for combinations they are all the same.

Examples: Evaluate

1. $\dfrac{10!}{8!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 = 90$

2. $P(8,3) = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = 336$

3. $C(11,9) = \dfrac{11!}{9!(11-9)!} = \dfrac{11!}{9!\,2!} = \dfrac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot2\cdot1} = \dfrac{11\cdot10}{2\cdot1} = 55$

4. $P(11,9) = \dfrac{11!}{(11-9)!} = \dfrac{11!}{2!} = 11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3 = 19{,}958{,}400$

5. $C(10,1) = \dfrac{10!}{1!(10-1)!} = \dfrac{10!}{1!\,9!} = 10$

Example: How many three letter sequences use the letters of b, o, g, e, y at most once each?

permutation

$P(5,3) = 60$

have want

Example: How many three letter sets use the letters of b, o, g, e, y at most once?

combination

$C(5,3) = 10$

have want

Example: A bag contains three red marbles, two green ones, one lavender one, two yellows and two orange marbles.

total = 10 marbles

1. How many possible sets of four marbles are there?

$$C(10,4) = \frac{10!}{4!\cdot 6!} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$$

2. How many sets of four marbles include all the red ones?

$$C(3,3)\cdot C(7,1) = \frac{3!}{3!\,0!}\cdot\frac{7!}{1!\,6!} = 1\cdot 7 = 7$$

have 3 red

want 3 red

have 7 others

want 1 of them

"have" must total 10, "want" must total 4

3. How many sets of four marbles include none of the red ones?

$$C(3,0)\cdot C(7,4) = \frac{3!}{0!\,3!}\cdot\frac{7!}{4!\,3!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$$

have 3 red

want no red

4. How many sets of four marbles include one of each color other than lavender?

$$C(3,1)\cdot C(2,1)\cdot C(1,0)\cdot C(2,1)\cdot C(2,1)$$

$$= 3\cdot 2\cdot 1\cdot 2\cdot 2$$

$$= 24$$