1.5 Infinite Limits

Definition of Infinite Limits – Let *f* be a function that is defined at every real number in some open interval containing *c* (except possibly at *c* itself). The statement $\lim_{x\to c} f(x)=\infty$ means that for each *M > 0* there exists a δ > 0 such that $f(x)>M$ whenever $0<|x-c|<\delta$. Similarly, the statement $\lim_{x\to c} f(x)=-\infty$ means that for each *N < 0* there exists δ > 0 such that $f(x)<N$ whenever $0<|x-c|<\delta$. To define the infinite limit from the left, replace $0<|x-c|<\delta$ by $c-\delta<x<c$. To define the infinite limit from the right, replace $0<|x-c|<\delta$ by $c<x<c+\delta$.

Note 1: Having a limit equal to infinity does NOT mean that the limit exists. In fact, it means the limit is unbounded and therefore does not exist.

Note 2: In WebAssign, if the limit is ∞ you should enter that for your answer and not DNE. There does not seem to be a great consistency in this however.

Definition of Vertical Asymptote – If *f(x)* approaches infinity (positive or negative) as *x* approaches *c* from the right or the left, then the line *x = c* is a vertical asymptote of the graph of *f(x)*.

Theorem on Vertical Asymptotes – Let *f* and *g* be continuous on an open interval containing *c*. If $f(c)\neq 0$, $g(c)=0$, and there exists and open interval containing *c* such that $g(x)\neq 0$ for all $x\neq c$ in the interval, then the graph of the function given by $h(x)=\dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x=c$.

Examples: Find the vertical asymptotes, if any.

1. $f(x)=\dfrac{-4x}{x^2+4}$ ← numerator is zero when $x=0$

← denominator is never zero for real x

Therefore, no vertical asymptotes.

2. $h(s)=\dfrac{2s-3}{s^2-25}$

numerator is zero at $s=\frac{3}{2}$

denom. is zero at $s=\pm 5$

Vertical asymptotes are $s=-5$ and $s=+5$

Note: vertical asymptotes are always equations of lines

3. $g(x)=\dfrac{x^3+1}{x+1} = \dfrac{(x+1)(x^2+x+1)}{x+1}$

Both numerator and denominator are zero at $x=-1$. This is not a vertical asymptote.

4. $h(t)=\dfrac{t^2-2t}{t^4-16} = \dfrac{t(t-2)}{(t-2)(t+2)(t^2+4)}$

$t=2$ is not a vertical asymptote as it is a removable discontinuity.

However, $t=-2$ is a vertical asymptote.

5. $f(x)=\sec \pi x = \dfrac{1}{\cos(\pi x)}$

numerator is never zero.

denominator is zero when $\pi x = \frac{\pi}{2}+\pi n$, where n is an arbitrary integer.

This means $f(x)=\sec(\pi x)$ has vertical asymptotes at $x=\frac{1}{2}+n$, for all integers n. (Solve $\pi x = \frac{\pi}{2}+\pi n$ for x).

Examples: Find the limit, if it exists.

1. $\lim_{x\to 1^+} \dfrac{2+x}{1-x}$

right of 1

Tables are great if you don't have a graph.

x	1	1.001	1.01	1.1
$f(x)$	?	-3001	-301	-31

$f(x)$ gets large and negative as $x\to 1^+$

so $\lim_{x\to 1^+} \dfrac{2+x}{1-x} = -\infty$

2. $\lim_{x \to 4^+} \frac{x^2}{x^2+16}$

x	4.1	4.01	4.001	4.0001	4
$f(x)$	0.5123	0.5012	0.5001	0.5000	?

$\lim_{x \to 4^+} \frac{x^2}{x^2+16} = \frac{1}{2}$

This could have been found by evaluating first. Much easier (and quicker) that way.

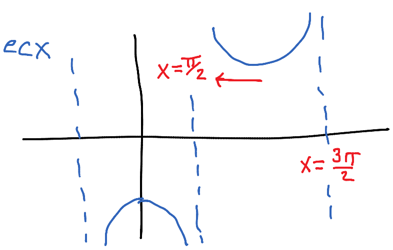
3. $\lim_{x \to (\pi/2)^+} \frac{-2}{\cos x}$

Since $\cos \frac{\pi}{2} = 0$, we should use a table

x	$\pi/2$	what to choose?
$f(x)$	?	

Let's consider the graph instead

$f(x) = \frac{-2}{\cos x} = -2\sec x$



As $x \to \frac{\pi}{2}^+$, the graph of $f(x)$ approaches $+\infty$. Therefore $\lim_{x \to \frac{\pi}{2}^+} \frac{-2}{\cos x} = +\infty$

Example: A patrol car is parked 50 feet from a long warehouse. The revolving light on top of the car turns at a rate of ½ revolution per second. The rate at which the light beam moves along the wall is $r = 50\pi \sec^2 \theta$ ft/sec.

a) Find the rate r when θ is π/6.

$r = 50\pi \sec^2\left(\frac{\pi}{6}\right)$
$= 50\pi \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = 50\pi \cdot \frac{4}{3} = \frac{200\pi}{3} \approx 209.4$ ft/sec

exact   approximation

b) Find the rate r when θ is π/3.

$r = 50\pi \sec^2\left(\frac{\pi}{3}\right) = 50\pi \cdot \frac{1}{\left(\frac{1}{2}\right)^2} = 200\pi \approx 628.3$ ft/sec

c) Find the limit of r as $\theta \to (\pi/2)^-$

We know that r will be $\pm\infty$ at $\pi/2$ but we need to determine which one it is. Notice in parts (a) and (b) that r is increasing in a positive direction. This leads to the conclusion $\lim_{x \to \frac{\pi}{2}^-} r = +\infty$

Example: A 25 foot ladder is leaning against a house. If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of $r = \dfrac{25}{\sqrt{625 - x^2}}$ ft/sec. where x is the distance between the base of the ladder and the house.

a) Find the rate when x is 7 feet.

$$r = \frac{25}{\sqrt{625 - 7^2}} = \frac{25}{\sqrt{625-49}} = \frac{25}{\sqrt{576}} \approx 1.04 \text{ ft/sec}$$

exact approx.

b) Find the rate when x is 15 feet.

$$r = \frac{25}{\sqrt{625 - 15^2}} = \frac{25}{\sqrt{400}} = \frac{25}{20} = 1.25 \text{ ft/sec}$$

exact, no rounding used

c) Find the limit of *r* as $x \to 25^-$ Let's try a table ... although initial guess from the pattern is $+\infty$.

x	24.1	24.5	24.9	25
r	3.76	5.03	11.19	?

yes, it seems reasonable as the closer we get to 25 from the left, the higher the rate. $\lim_{x \to 25^-} r = +\infty$

Properties of Infinite Limits – Let *c* and *L* be real numbers and let *f* and *g* be functions such that $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$.

1. Sum or difference: $\lim_{x \to c} \left[f(x) \pm g(x) \right] = \infty$

2. Product: $\lim_{x \to c} \left[f(x) g(x) \right] = \infty, \quad L > 0$

   $\lim_{x \to c} \left[f(x) g(x) \right] = -\infty, \quad L < 0$

3. Quotient: $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of *f(x)* as *x* approaches *c* is -∞.