4.3 Riemann Sums and Definite Integrals

In 4.2 we focused on rectangles that all had the same width. But that isn't always the best way to go about breaking up an area, and many times it isn't even possible. In order to overcome this we need a definition that could allow the widths to be different sizes. For any selected partitioning of an interval we will call it Δ.

Definition of Riemann Sum – Let *f* be defined on the closed interval [a,b], and let Δ be a partition of [a,b] given by $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ where Δx_i is the width of the *i*th subinterval. If c_i is any point in the *i*th subinterval $[x_{i-1}, x_i]$, then the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i,\ x_{i-1} < c_i < x_i$ is called a Riemann sum of *f* of the partition Δ.

Fact: The width of the largest subinterval of a partition Δ is the norm of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is the regular and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}$$

Definition of Definite Integral – If *f* is defined on the closed interval [a, b] and the limit of Riemann sums over partitions Δ given by $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, then *f* is said to be integrable on [a, b] and the limit is denoted by $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$. The limit is called the definite integral of *f* from *a* to *b*. The number *a* is the lower limit of integration, and the number *b* is the upper limit of integration.

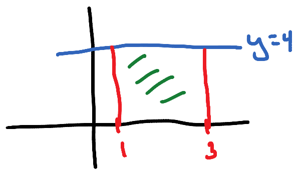
Theorem 4.4: Continuity Implies Integrability – If a function *f* is continuous on the closed interval [a, b], then *f* is integrable on [a, b]. That is, $\int_a^b f(x)\,dx$ exists.

Using the limit definition of a definite integral would have you consider changing your major to something that never uses math ever again. First you have to find a partition, then rewrite the integral as the limit of a sum, find the sum, take the limit, and finally you have an answer. There must be a better way! There is.

Theorem 4.5: The Definite Integral as the Area of a Region – If *f* is continuous and nonnegative on the closed interval [a, b], then the area of the region bounded by the graph of *f*, the x-axis, and the vertical lines *x* = *a* and *x* = *b* is given by $Area = \int_a^b f(x)\,dx$.

Examples: Sketch the region corresponding to each definite integral. Then evaluate using a geometric formula.

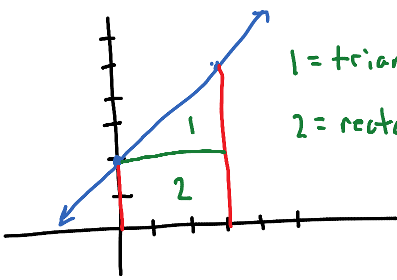
1. $\int_1^3 4dx = 2(4) = 8$



rectangle with $l=2$ and $w=4$

$A = lw$

2. $\int_0^3 (x+2)\,dx = \frac{1}{2}(3)(3) + 3(2) = \frac{9}{2} + 6 = 10.5$



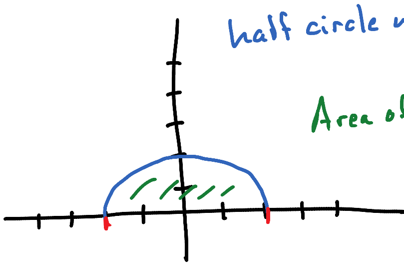
1 = triangle with $b=3$, $h=3$ $A = \frac{1}{2}bh$

2 = rectangle with $l=3$ $h=2$ $A = lw$

3. $\int_{-2}^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi(2)^4 = 2\pi$

half circle with $r=2$

Area of $\frac{1}{2}$ circle $= \frac{1}{2}\pi r^2$



Properties of Definite Integrals –

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on [a, b], then we define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

3. If f is integrable on the three closed intervals determined by *a, b*, and *c*, then
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Theorem 4.7: Properties of Definite Integrals – If f and g are integrable on [a, b] and k is a constant, then the functions kf and $f \pm g$ are integrable on [a, b], and

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Examples: Evaluate the integral using the following values. $\int_2^4 x^3\,dx = 60,\ \int_2^4 x\,dx = 6,\ \int_2^4 dx = 2$

1. $\int_2^2 x^3\,dx = 0$ By property 1, the width of the rectangle is 0

2. $\int_2^4 25\,dx = 25\int_2^4 dx = 25(2) = 50$

3. $\int_2^4 (x^3 + 4)\,dx = \int_2^4 x^3\,dx + 4\int_2^4 dx$

$= 60 + 4(2) = 68$

4. $\int_2^4 (10 + 4x - 3x^3)\,dx = 10\int_2^4 dx + 4\int_2^4 x\,dx - 3\int_2^4 x^3\,dx$

$= 10(2) + 4(6) - 3(60)$

$= 20 + 24 - 180 = -136$

Examples: Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, evaluate

1. $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

2. $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$