5.3 Inverse Functions

Definition of an Inverse Function – A function *g* is the inverse function of the function *f* if $f\left(g\left(x\right)\right)=x$ for each *x* in the domain of *g* and $g\left(f\left(x\right)\right)=x$ for each *x* in the domain of *f*. The function *g* is denoted by f^{-1} (read *f* inverse).

Examples: Show that *f* and *g* are inverse functions. → Show composition yields the identity in both directions

1. $f\left(x\right)=3-4x, \quad g\left(x\right)=\frac{3-x}{4}$

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) \\
&= f\left(\frac{3-x}{4}\right) \\
&= 3-\cancel{4}\left(\frac{3-x}{\cancel{4}}\right) \\
&= \cancel{3}-(\cancel{3}-x) \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= g(3-4x) \\
&= \frac{\cancel{3}-(\cancel{3}-4x)}{4} \\
&= \frac{\cancel{4}x}{\cancel{4}} = x
\end{aligned}
$$

2. $f\left(x\right)=1-x^{3}, \quad g\left(x\right)=\sqrt[3]{1-x}$

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) \\
&= f(\sqrt[3]{1-x}) \\
&= 1-(\sqrt[\cancel{3}]{1-x})^{\cancel{3}} \\
&= \cancel{1}-(\cancel{1}-x) \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= g(1-x^3) \\
&= \sqrt[3]{\cancel{1}-(\cancel{1}-x^3)} \\
&= \sqrt[\cancel{3}]{x^{\cancel{3}}} \\
&= x
\end{aligned}
$$

Reflective Property of Inverse Functions – The graph of *f* contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

The Existence of an Inverse Function:

1. A function has an inverse function if and only if it is one-to-one.

2. If *f* is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse.

Guidelines for Finding an Inverse Function –

1. Use the Existence Theorem to determine whether the function given by $y = f(x)$ has an inverse function.

2. Solve for *x* as a function of *y*: $x = g(y) = f^{-1}(y)$

3. Interchange *x* and *y*. The resulting equation is $y = f^{-1}(x)$.

4. Define the domain of f^{-1} as the range of *f*.

5. Verify with composition that the functions are indeed inverses.

Example: Find the inverse function.

1. $f(x) = 2x - 3$

$y = 2x - 3$

$y + 3 = 2x$

$\frac{y+3}{2} = x$

$y = \frac{x+3}{2}$

Is $f^{-1}(x) = \frac{x+3}{2}$

Use composition to verify these are inverses.

2. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

$y = \sqrt{4 - x^2}$

$y^2 = 4 - x^2$

$y^2 - 4 = -x^2$

$4 - y^2 = x^2$

$\pm\sqrt{4 - y^2} = x$

Based on original domain [0,2] of f, the range of f^{-1} will be [0,2]. This allows us to choose positive or negative root.

$y = \sqrt{4 - x^2}$

$f^{-1}(x) = \sqrt{4 - x^2}$

Check using composition

3. $f(x)=\frac{x+2}{x}$

$y=\frac{x+2}{x}$

$xy=x+2$

$xy-x=2$

$x(y-1)=2$

$x=\frac{2}{y-1}$

$y=\frac{2}{x-1}$

$f^{-1}(x)=\frac{2}{x-1}$

Check $(f\circ f^{-1})(x)=f(f^{-1}(x))$

$=f\left(\frac{2}{x-1}\right)$

$=\dfrac{\frac{2}{x-1}+\frac{2}{1}\cdot\frac{x-1}{x-1}}{\frac{2}{x-1}}$

$=\dfrac{\frac{2}{x-1}+\frac{2x-2}{x-1}}{\frac{2}{x-1}}=\frac{2x}{x-1}\cdot\frac{x-1}{2}=x$

Verify the other...

But how do we know for sure that all of these functions are one-to-one? We could graph them. Or we could use calculus. If a function has a first derivative that is always positive, or always negative, then we know it is monotonic.

Example: Show that $f(x)=\frac{4}{x^2}$, $(0,\infty)$ is strictly monotonic on the interval and therefore has an inverse function on that interval.

Rewrite: $f(x)=4x^{-2}$

$f'(x)=-8x^{-3}=\frac{-8}{x^3}$

numerator is always negative

on $(0,\infty)$ denominator is always positive

Analysis: A negative divided by a positive will always be negative so this function is decreasing on $(0,\infty)$.

The Derivative of an Inverse Function – Let *f* be a function that is differentiable on an interval *I*. If *f* has an inverse function *g*, then *g* is differentiable at any *x* for which $f'(g(x))\neq 0$. Moreover,

$$g'(x)=\frac{1}{f'(g(x))},\quad f'(g(x))\neq 0$$

Examples: Verify that *f* has an inverse. Then use the function *f* and the given real number *a* to find $\left(f^{-1}\right)'(a)$.

1. $f(x) = 5 - 2x^3, \quad a = 7$

$f'(x) = -6x^2$

analysis: x^2 is always positive
so $-6x^2$ is always negative
an inverse exists

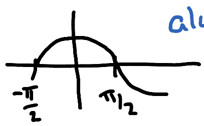
We want 7 to be input of inverse, so it is output of f. Solving $7 = 5 - 2x^3$ we get $x = 1$

If $f(1) = 7$, then $f^{-1}(7) = 1$

$$\left(f^{-1}\right)'(7) = \frac{1}{f'(1)} = \frac{1}{-6(1)^2} = \frac{1}{-6}$$

2. $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad a = \frac{1}{2}$

$f'(x) = \cos x$



always positive on the interval

$\sin x = \frac{1}{2}$

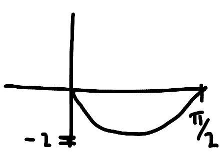
$x = \pi/6$

$f(\frac{\pi}{6}) = \frac{1}{2}, \; f^{-1}(\frac{1}{2}) = \pi/6$

$$\left(f^{-1}\right)'\left(\tfrac{1}{2}\right) = \frac{1}{f'(\pi/6)} = \frac{1}{\cos \pi/6} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

3. $f(x) = \cos 2x, \quad 0 \le x \le \frac{\pi}{2}, \quad a = 1$

$f'(x) = -2\sin 2x$



always negative on the interval

$\cos 2x = 1$
$2x = 0$
$x = 0$

$f(0) = 1, \; f^{-1}(1) = 0$

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{-2\sin(0)} = \frac{1}{0} = \text{undefined}$$