5.5 Bases Other than *e* and Applications

Definition – If *a* is a positive real number not equal to 1 and *x* is any real number, then the exponential function to the base *a* is denoted by $a^x$ and is defined by $a^x = e^{(\ln a)x}$. If *a* = *1*, then $y = 1^x = 1$ is a constant function.

x

Definition – If *a* is a positive real number not equal to 1 and *x* is any positive real number, then the logarithmic function to the base *a* is denoted by $\log_a x$ and is defined as $\log_a x = \frac{1}{\ln a}\ln x$.

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These are not your normal definitions.

Properties of Inverse Functions –

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x, \; for \;\; x > 0$
3. $\log_a a^x = x, \quad for\, all \;\; x$

Examples: Evaluate

1. $\log_2 \frac{1}{8} = x$

$2^x = \frac{1}{8}$

$2^x = \frac{1}{2^3} = 2^{-3}$

$x = -3$

$\log_2\left(\frac{1}{8}\right) = -3$

2. $\log_7 1 = y$

$7^y = 1$

y must be 0

$\log_7 1 = 0$

3. $\log_a \frac{1}{a} = b$

$a^b = \frac{1}{a} = a^{-1}$

$b = -1$ so $\log_a \frac{1}{a} = -1$

Examples: Solve

1. $\log_3 \frac{1}{81} = x$

$3^x = \frac{1}{81} = 3^{-4}$

$x = -4$

2. $\log_b 27 = 3$

$b^3 = 27 = 3^3$

so $b = 3$

3. $3x+5=\log_2 64$

$2^6=64$

$3x+5=6$

$3x=1$

$x=\frac{1}{3}$

4. $\frac{3\left(5^{x-1}\right)}{3}=\frac{86}{3}$

$5^{x-1}=\frac{86}{3}$

$x-1=\log_5\left(86/3\right)$

$x=1+\log_5(86/3)\approx 3.085$

Derivatives for bases other than *e* – Let *a* be a positive real number, not equal to 1, and let *u* be a differentiable function of *x*.

1. $\frac{d}{dx}\left[a^x\right]=(\ln a)a^x$

2. $\frac{d}{dx}\left[a^u\right]=(\ln a)a^u\frac{du}{dx}$

3. $\frac{d}{dx}\left[\log_a x\right]=\frac{1}{(\ln a)x}$

4. $\frac{d}{dx}\left[\log_a u\right]=\frac{1}{(\ln a)u}\frac{du}{dx}=\frac{u'}{(\ln a)u}$

Examples: Find the derivative. Hint: sometimes using log properties first makes things easier.

1. $f(x)=3^{2x}$

$u=2x$

$\frac{du}{dx}=2$

$f'(x)=(\ln 3)3^{2x}\cdot 2$

$f'(x)=(2\ln 3)3^{2x}$

2. $y=7^{2x-1}$

$u=2x-1$

$\frac{du}{dx}=2$

$y'=(2\ln 7)7^{2x-1}$

3. $y=x\left(6^{-2x}\right)$

product rule

$y'=x(-2\ln 6)6^{-2x}+\left(6^{-2x}\right)(1)$

$y'=6^{-2x}(-2x\ln 6+1)$

4. $y=\log_3\left(x^2-3x\right)$

$y'=\frac{1}{(\ln 3)(x^2-3x)}\cdot(2x-3)=\frac{2x-3}{(\ln 3)(x^2-3x)}$

you rarely see the base of 10 written

5. $y = \log_{10} \frac{x^2-1}{x}$

Rewrite: $y = \log(x^2-1) - \log x$

$$y' = \frac{2x}{(\ln 10)(x^2-1)} - \frac{1}{(\ln 10)x}$$

Example: Find an equation of the tangent line to the graph of $y = 5^{x-2}$ at the point (2,1).

$y' = (\ln 5)5^{x-2}$

$m = (\ln 5)(5^{2-2}) = \ln 5$

$y - 1 = \ln 5(x-2)$

$y - 1 = x\ln 5 - 2\ln 5$

$y = (\ln 5)x - 2\ln 5 + 1$

Example: Use logarithmic differentiation to find dy/dx for $y = x^{x-1}$.

$\ln y = \ln x^{x-1}$

$\ln y = (x-1)\ln x$

$\ln y = x\ln x - \ln x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x - \frac{1}{x}$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x - \frac{1}{x}$

$\frac{dy}{dx} = \left(1 + \ln x - \frac{1}{x}\right)\left(x^{x-1}\right)$

leave it...

Integration Formula - $\int a^x dx = \left(\frac{1}{\ln a}\right)a^x + C$

derivative multiplies, integral divides

Examples: Find the integral

1. $\int (x^3 + 3^x)dx$

$= \int x^3 dx + \int 3^x dx = \frac{x^4}{4} + \frac{1}{\ln 3}\cdot 3^x + C = \frac{x^4}{4} + \frac{3^x}{\ln 3} + C$

2. $\int (3-x)7^{(3-x)^2} dx$

Let $u = (3-x)^2$

$du = 2(3-x)(-1)\,dx$

$-\frac{1}{2}du = (3-x)dx$

$-\frac{1}{2}\int 7^u du = -\frac{1}{2}\cdot\frac{1}{\ln 7}\cdot 7^u + C = -\frac{7^{(3-x)^2}}{2\ln 7} + C$

3. $\int 2^{\sin x} \cos x dx = \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{\sin x}}{\ln 2} + C$

Let $u = \sin x$

$du = \cos x dx$

Example: Find the area of the region bounded by $y = 3^{\cos x} \sin x,\ y = 0,\ x = 0,\ x = \pi$

integral

function

limits of integration

$$\int_0^{\pi} 3^{\cos x} \sin x dx = -\int_1^{-1} 3^u du = -\frac{3^u}{\ln 3}\Big|_1^{-1} = \frac{-3^{-1}}{\ln 3} - \left(\frac{-3^1}{\ln 3}\right) = \frac{3}{\ln 3} - \frac{1}{3\ln 3}$$

$u = \cos x$ $\quad$ $x = 0,\ u = 1$

$du = -\sin x dx$ $\quad$ $x = \pi \quad u = -1$

$-du = \sin x dx$

Theorem 5.15 – A limit involving *e*: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$.

I give you this theorem as the development of the constant *e* from financial formulas. It is a good thing to know, but not essential.