5.6 Inverse Trigonometric Functions: Differentiation

Definitions of inverse trig functions – Note the restricted domains of the original trigonometric functions so that the inverse is also a function.

Function	Domain	Range
$y = \arcsin x$ *iff* $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ QI + QIV
$y = \arccos x$ *iff* $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$ QI + QII
$y = \arctan x$ *iff* $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$ QI + QIV
$y = arc\cot x$ *iff* $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$ QI + QII
$y = arc\sec x$ *iff* $\sec y = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$ QI + QII
$y = arc\csc x$ *iff* $\csc y = x$	$\lvert x\rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$ QI + QIV

Examples: Evaluate the expression without using a calculator.

1. $\arcsin 0 = y$ if $\sin y = 0$.

   $y = 0, \pi, 2\pi, \ldots$

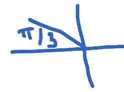
   Restricted so only answer is $\boxed{y = 0}$

2. $\arccos 1 = y$

   if $\cos y = 1$. Thus $y = 0$

3. $arc\cot\left(-\sqrt{3}\right) = y$ if $\cot y = -\sqrt{3} = \frac{-\sqrt{3}}{1} = \frac{-\sqrt{3}/2}{1/2}$

   This happens at $\pi/3$ reference angles but with sine and Cosine having opposite signs $\Rightarrow$ QII or QIV



   $\boxed{y = \frac{2\pi}{3}}$

4. $\tan\left(\arccos\frac{\sqrt{2}}{2}\right)$

   Let $y = \arccos\frac{\sqrt{2}}{2}$

   then $\cos y = \frac{\sqrt{2}}{2}$

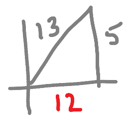
   and $y = \pi/4$

   $\tan \pi/4 = 1$

5. $\cos\left(\arcsin\frac{5}{13}\right) = \cos y = \boxed{\frac{12}{13}}$

Let $y = \arcsin\frac{5}{13}$

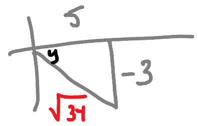
then $\sin y = \frac{5}{13}$



6. $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \sec y = \boxed{\frac{\sqrt{34}}{5}}$

Let $y = \arctan(-3/5)$

Then $\tan y = -\frac{3}{5}$



Derivatives of Inverse Trig Functions – Let *u* be a differentiable function of *x*.

Know these three

pay attention to the form of each

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[arc\cot u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[arc\sec u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[arc\csc u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

these are just the negatives of the main 3

Examples: Find the derivative of the function.

1. $f(t) = \arcsin t^2$

Let $u = t^2$

$u' = 2t$

$$f'(t) = \frac{2t}{\sqrt{1-(t^2)^2}} = \frac{2t}{\sqrt{1-t^4}}$$

2. $f(x) = arc\sec 2x$

Let $u = 2x$

$u' = 2$

$$f'(x) = \frac{2}{|2x|\sqrt{(2x)^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$$

3. $f(x) = \arctan\sqrt{x}$

$u = \sqrt{x}$

$u' = \frac{1}{2\sqrt{x}}$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{1+(\sqrt{x})^2} = \frac{1}{2\sqrt{x}(1+x)}$$

4. $h(x) = x^2 \arctan 5x$

product rule

$$h'(x) = x^2 \frac{5}{1+25x^2} + 2x \arctan 5x$$

$y = \arctan 5x$

$u = 5x$

$u' = 5$

$y' = \frac{5}{1+(5x)^2}$

5. $f(x) = \arcsin x + \arccos x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$$