Finding Limits Graphically and Numerically

By Tuesday J. Johnson

Suggested Review Topics

- Algebra skills reviews suggested:
  - Evaluating functions
  - Graphing functions
  - Working with inequalities
  - Working with absolute values
  - Conjugates and difference of squares
- Trigonometric skills reviews suggested:
  - None

Calculus
Limits and Their Properties

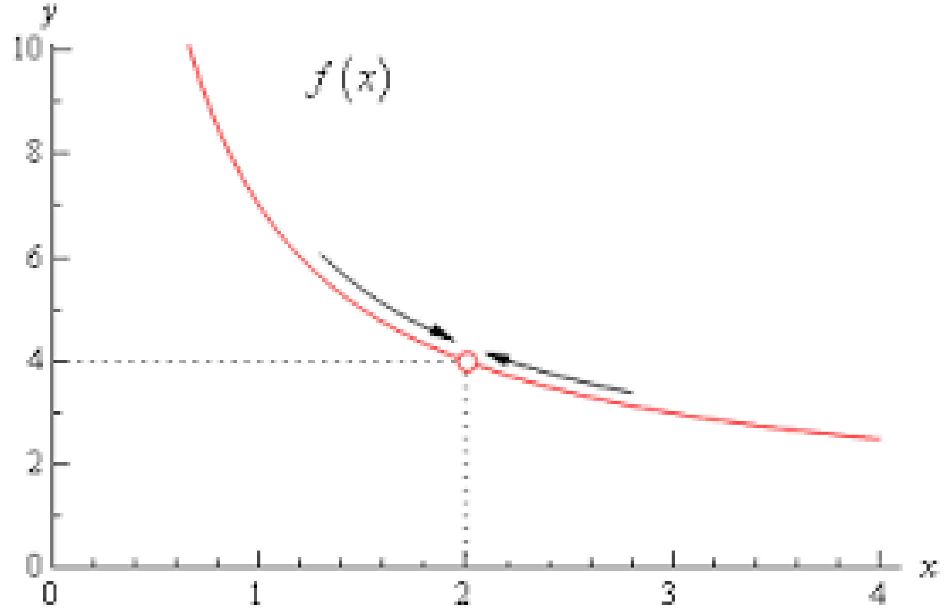
Finding Limits Graphically and Numerically

Limits

- Graphing functions seems pretty straightforward for functions that have a domain of all real numbers. We choose a few domain points, find the corresponding range values, then plot and join with a smooth curve.
- When the domain has exclusions we need to determine what is going on at or near these values. In order to do this we use what is called a limit.

Informal Definition

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the limit of $f(x)$ as x approaches c is L.



Informal Definition

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the limit of $f(x)$ as x approaches c is L.

- The limit is written as
$$\lim_{x \to c} f(x) = L$$

Examples: Use a table of values to estimate the limit numerically.

1. $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-4}$

2. $\lim\limits_{x \to -5} \dfrac{\sqrt{4-x}-3}{x+5}$

3. $\lim\limits_{x \to 4} \dfrac{\frac{x}{x+1}-\frac{4}{5}}{x-4}$

4. $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$

Examples: Use a table of values to estimate the limit numerically.

1. $\lim_{x \to 2} \frac{x-2}{x^2-4}$

We will choose values close to 2 from both less than 2 (below 2) and greater than 2 (above 2).

x	1.9	1.99	1.999	2
$\frac{(x-2)}{(x^2-4)}$				???
x	2.1	2.01	2.001	2
$\frac{(x-2)}{(x^2-4)}$				???

Examples: Use a table of values to estimate the limit numerically.

1. $\lim\limits_{x \to 2} \frac{x-2}{x^2-4}$

Replace x with each value to fill in the table for values below 2.

x	1.9	1.99	1.999	2
$\frac{(x-2)}{(x^2-4)}$	0.2564	0.2506	0.2501	???

Examples: Use a table of values to estimate the limit numerically.

1. $\lim_{x \to 2} \frac{x-2}{x^2-4}$

Replace x with each value to fill in the table as well for values above 2.

x	1.9	1.99	1.999	2
$\frac{(x-2)}{(x^2-4)}$	0.2564	0.2506	0.2501	???
x	2.1	2.01	2.001	2
$\frac{(x-2)}{(x^2-4)}$	0.2439	0.2494	0.2499	???

Examples: Use a table of values to estimate the limit numerically.

$$1.\ \lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{4} = 0.25$$

Now make an educated guess for the limit.

x	1.9	1.99	1.999	2
$\frac{x-2}{x^2-4}$	0.2564	0.2506	0.2501	**0.25**
x	2.1	2.01	2.001	2
$\frac{x-2}{x^2-4}$	0.2439	0.2494	0.2499	**0.25**

Examples: Use a table of values to estimate the limit numerically.

2. $\lim_{x \to -5} \frac{\sqrt{4-x}-3}{x+5}$

Replace x with values both above and below -5.

x	-5.1	-5.01	-5.001	-5
$\frac{(\sqrt{(4-x)}-3)}{(x+5)}$	-0.1662	-0.1666	-0.1667	???
x	-4.9	-4.99	-4.999	-5
$\frac{(\sqrt{(4-x)}-3)}{(x+5)}$	-0.1671	-0.1667	-0.1667	???

Examples: Use a table of values to estimate the limit numerically.

2. $\lim_{x \to -5} \frac{\sqrt{4-x}-3}{x+5}$

Make an educated guess as to the limit.

x	-5.1	-5.01	-5.001	-5
$\frac{(\sqrt{(4-x)}-3)}{(x+5)}$	-0.1662	-0.1666	-0.1667	**-0.166…**
x	-4.9	-4.99	-4.999	-5
$\frac{(\sqrt{(4-x)}-3)}{(x+5)}$	-0.1671	-0.1667	-0.1667	**-0.166…**

Examples: Use a table of values to estimate the limit numerically.

2. $\lim_{x \to -5} \frac{\sqrt{4-x}-3}{x+5} = -\frac{1}{6}$

x	-5.1	-5.01	-5.001	-5
$\frac{(\sqrt{(4-x)}-3)}{(x+5)}$	-0.1662	-0.1666	-0.1667	**-0.166…**
x	-4.9	-4.99	-4.999	-5
$\frac{(\sqrt{(4-x)}-3)}{(x+5)}$	-0.1671	-0.1667	-0.1667	**-0.166…**

- It looks like $f(x)$ approaches $-0.16666\ldots$ but we will ALWAYS use exact values unless specified otherwise.

Examples: Use a table of values to estimate the limit numerically.

3. $\lim_{x \to 4} \frac{\frac{x}{x+1} - \frac{4}{5}}{x-4}$

x	3.9	3.99	3.999	4
$\frac{\left(\frac{x}{(x+1)} - \frac{4}{5}\right)}{(x-4)}$	0.0408	0.0401	0.0400	???
x	4.1	4.01	4.001	4
$\frac{\left(\frac{x}{(x+1)} - \frac{4}{5}\right)}{(x-4)}$	0.0392	0.0399	0.0400	???

Examples: Use a table of values to estimate the limit numerically.

3. $\lim\limits_{x \to 4} \dfrac{\frac{x}{x+1} - \frac{4}{5}}{x-4} = 0.04 = \dfrac{1}{25}$

x	3.9	3.99	3.999	4
$\dfrac{\left(\frac{x}{(x+1)} - \frac{4}{5}\right)}{(x-4)}$	0.0408	0.0401	0.0400	**0.04**
x	4.1	4.01	4.001	4
$\dfrac{\left(\frac{x}{(x+1)} - \frac{4}{5}\right)}{(x-4)}$	0.0392	0.0399	0.0400	**0.04**

Examples: Use a table of values to estimate the limit numerically.

4. $\lim_{x \to 0} \frac{\sin 4x}{x} = 4$

- ALWAYS use radians in Calculus! Take your calculator off degree mode now and keep it there. Unless a specific problem states otherwise.

x	-0.1	-0.01	-0.001	0
$\frac{(\sin(4x))}{x}$	3.8942	3.9989	4.0000	**4**
x	0.1	0.01	0.001	0
$\frac{(\sin(4x))}{x}$	3.8942	3.9989	4.0000	**4**

Examples: Use the graph to find the limit.

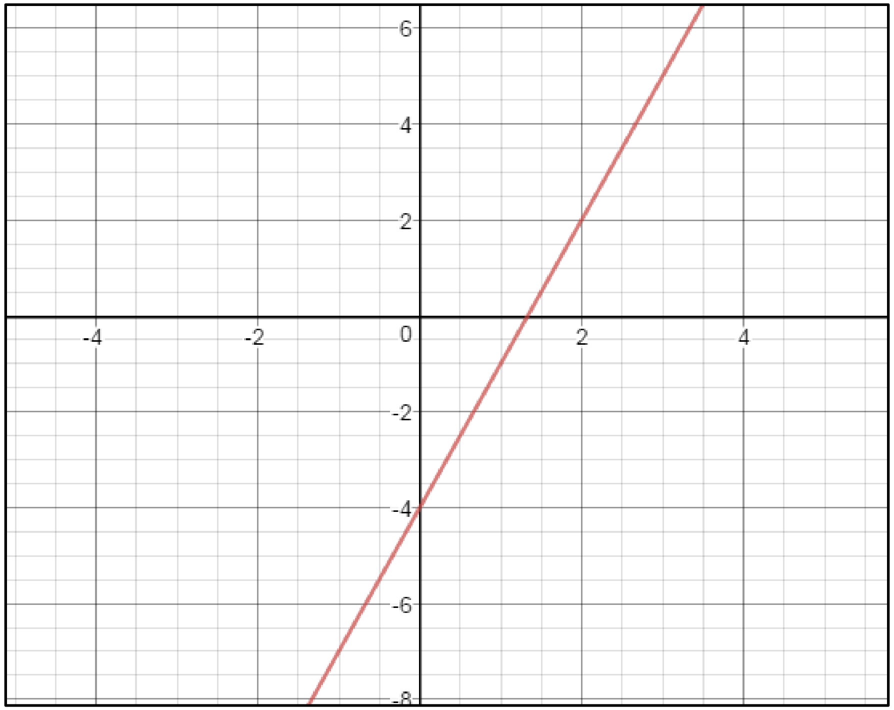
1. $\lim\limits_{x \to 2} (3x - 4)$

2. $\lim\limits_{x \to -2} \frac{|x+2|}{x+2}$

3. $\lim\limits_{x \to 1} \frac{5}{x-1}$

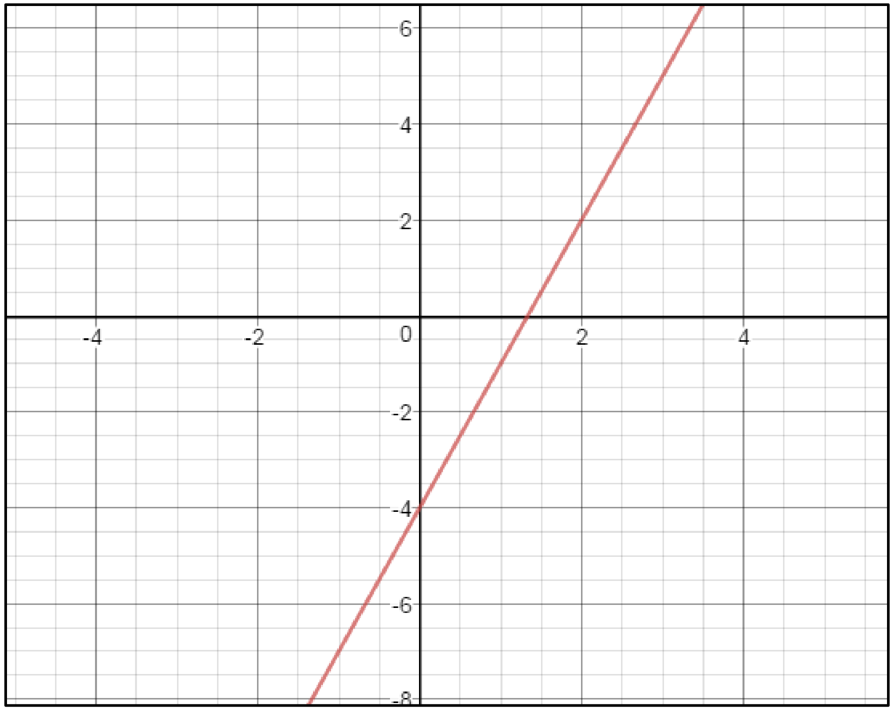
Examples: Use the graph to find the limit.

1. $\lim_{x \to 2}(3x - 4)$



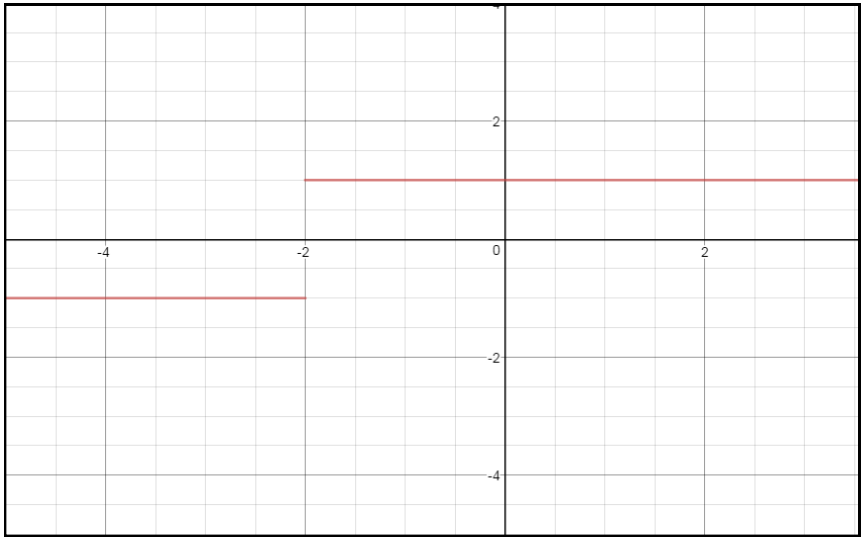
Examples: Use the graph to find the limit.

1. $\lim_{x \to 2}(3x - 4) = 2$



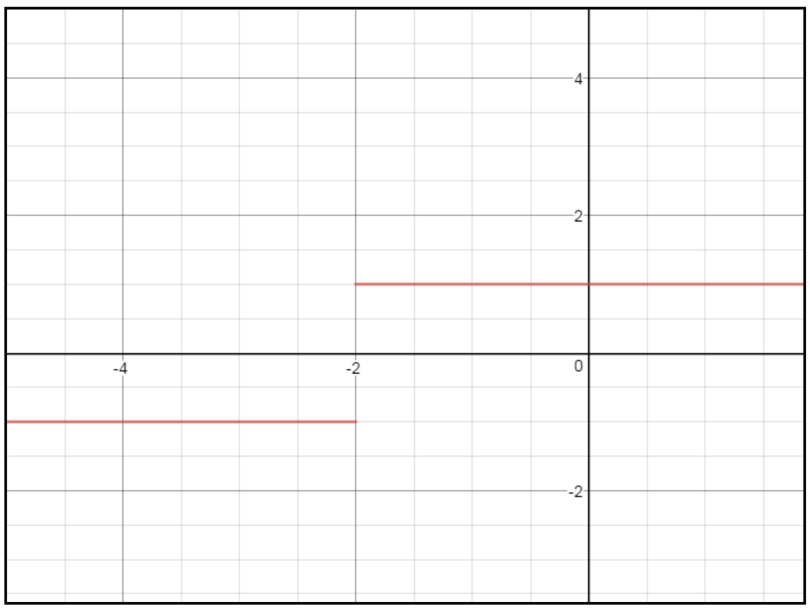
Examples: Use the graph to find the limit.

2. $\lim_{x \to -2} \frac{|x+2|}{x+2}$
   - Notice that the numerator and denominator are the same for values larger than -2 and only have opposite value for x values smaller than -2.



Examples: Use the graph to find the limit.

2. $\lim_{x \to -2} \frac{|x+2|}{x+2}$

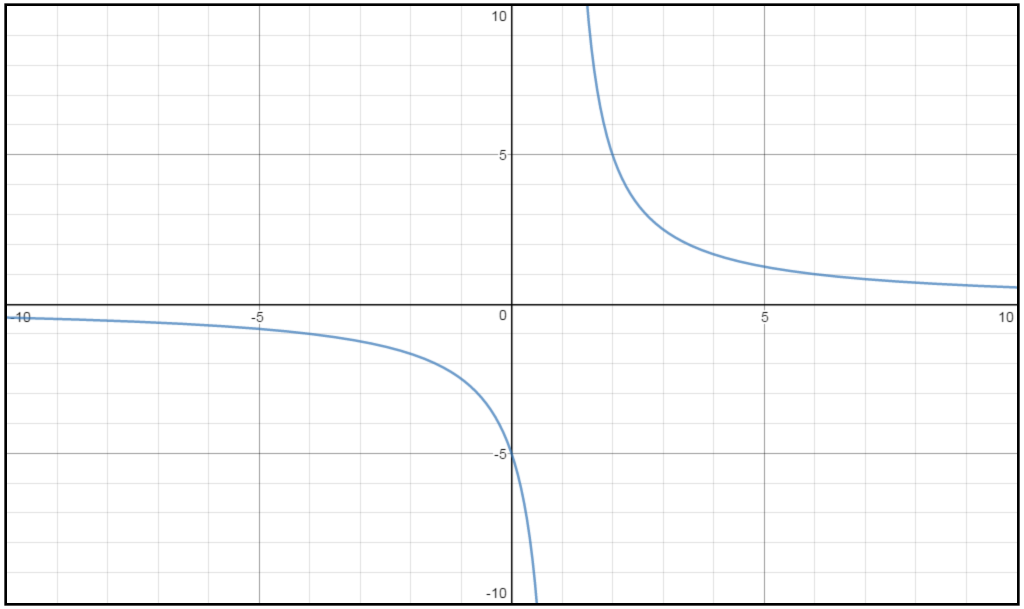


As $x \to -2$ from the right, the output is 1. However, as we approach from the left the function value goes to -1. These values do not match so the limit does not exist (DNE).

Examples: Use the graph to find the limit.

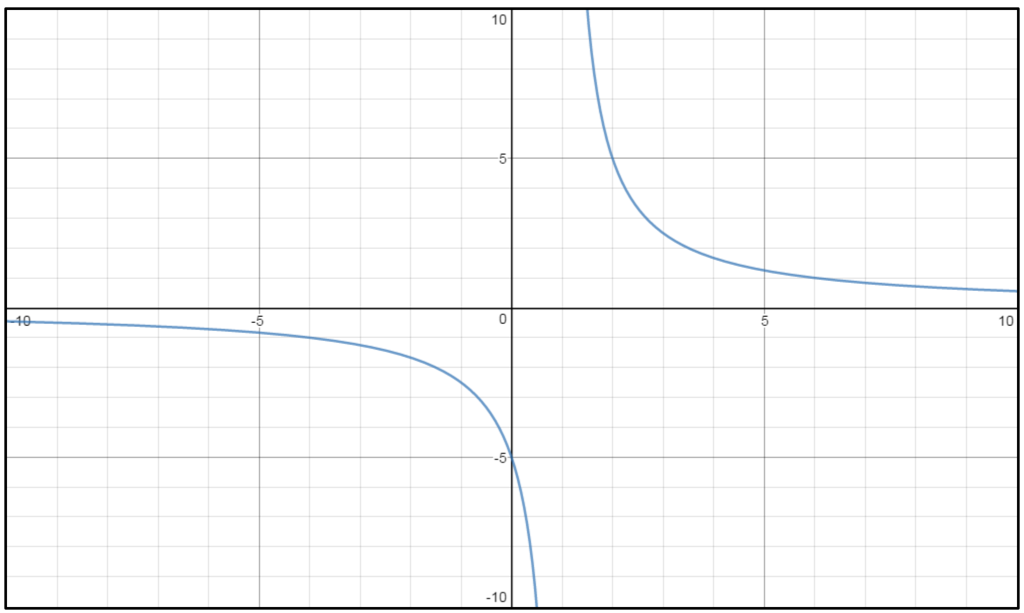
2. $\lim_{x \to 1} \frac{5}{x-1}$

– Remembering translations from Pre-Calculus, we can graph this:



Examples: Use the graph to find the limit.

3. $\lim_{x \to 1} \frac{5}{x-1}$



As $x \to 1$, the graph seems to "blow up" to both positive and negative infinity. This also means that the limit does not exist (DNE).

Common Types of Behavior Associated with Nonexistence of a Limit

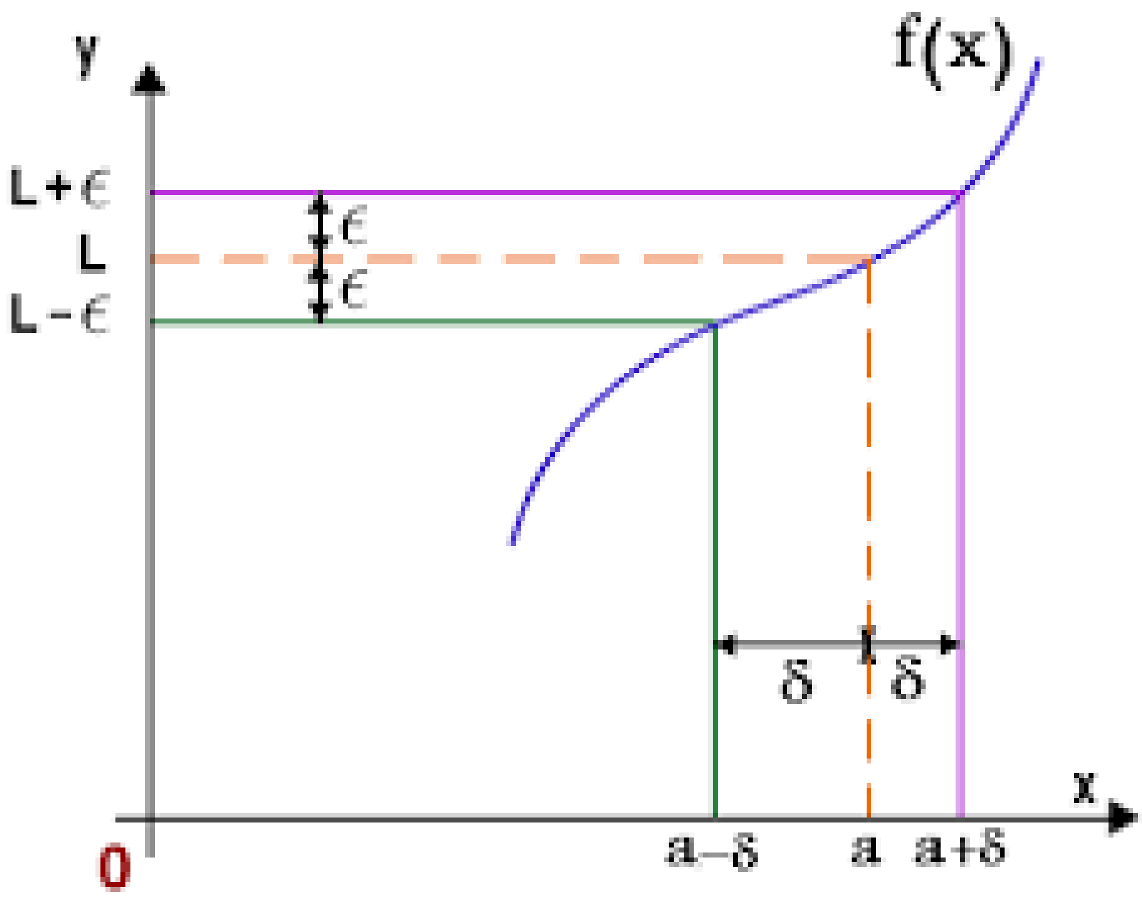
1. The function approaches a different number from the right side than it approaches from the left side.
2. The function increases (or decreases) without bound as x approaches c. That is, the function goes to infinity.
3. The function oscillates between two fixed values. This frequently occurs with trigonometric functions.

Formal Limit Definition

Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

- This is known as the $\varepsilon - \delta$ definition of a limit.

The $\varepsilon - \delta$ definition of a limit graphically.



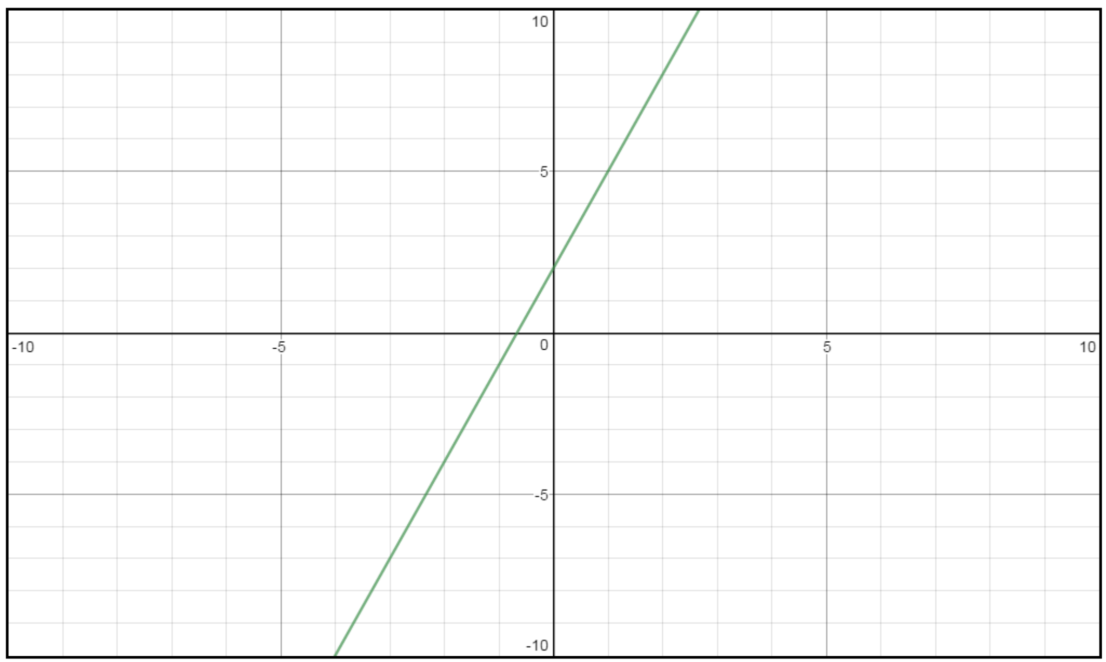
Examples: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

1. $\lim_{x\to 2}(3x + 2)$

2. $\lim_{x\to 5}(x^2 + 4)$

Examples: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

1. $\lim_{x \to 2}(3x + 2) = 8$



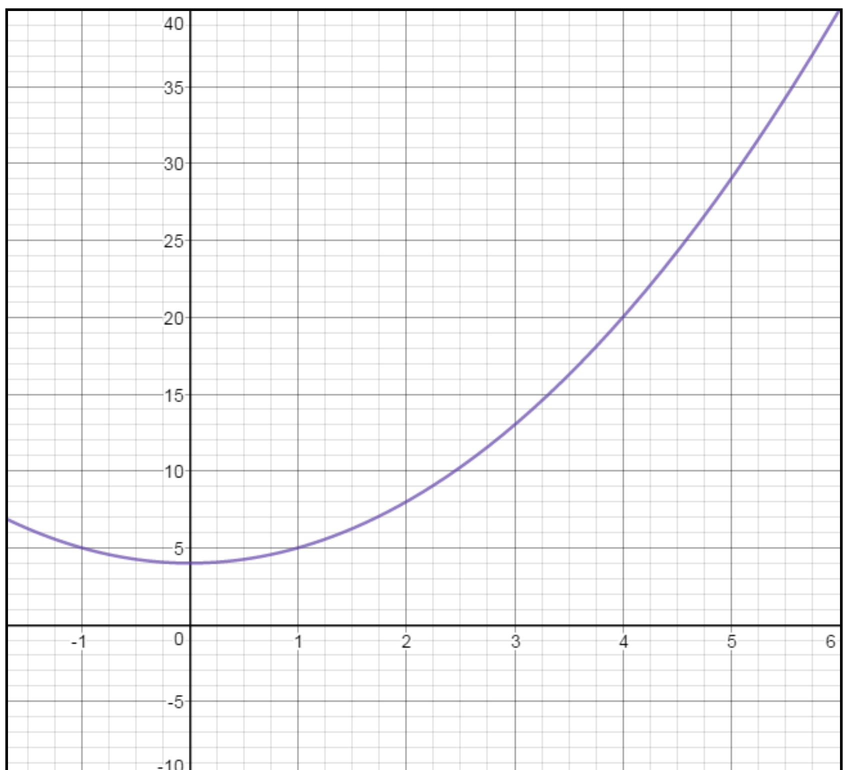
Examples: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

1. $\lim_{x \to 2}(3x + 2) = 8$

Start with $|f(x) - L| < 0.01$ and make your substitutions to get $|3x + 2 - 8| < 0.01$ or $|3x - 6| < 0.01$. We can factor to get $3|x - 2| < 0.01$ and finally divide to get $|x - 2| < {}^{0.01}/_{3}$. That is, let $\delta = {}^{0.01}/_{3}$.

Examples: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

2. $\lim_{x\to 5}(x^2 + 4) = 29$



Examples: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

2. $\lim\limits_{x \to 5}(x^2 + 4) = 29$

Goal: We want to get to $|x - 5| < \delta$ while starting with $|f(x) - 29| < 0.01$

$$|f(x) - L| < 0.01$$
$$|x^2 + 4 - 29| < 0.01$$
$$|x^2 - 25| < 0.01$$
$$|x - 5||x + 5| < 0.01$$

Examples: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

2. $\lim_{x \to 5} (x^2 + 4) = 29$

$$|x - 5||x + 5| < 0.01$$

Our goal is to isolate $|x - 5|$. We can divide by $|x + 5|$ as long as it is not zero. Near $x = 5$, this is not zero and it is at most 11. {Why 11? Near 5 is 6, and $6 + 5 = 11$.

$$|x - 5| < \frac{0.01}{|x + 5|}$$

$$|x - 5| < \frac{0.01}{11}$$

Therefore $\delta = {}^{0.01}/_{11}$ works.

Examples: Find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

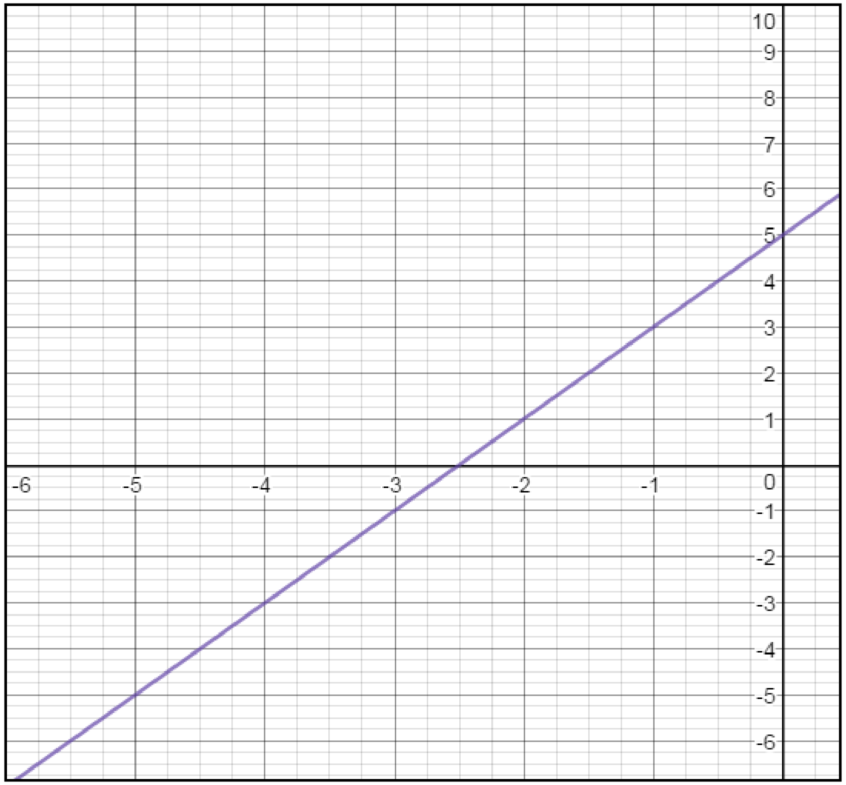
1. $\lim\limits_{x \to -3} (2x + 5)$

2. $\lim\limits_{x \to 4} \sqrt{x}$

3. $\lim\limits_{x \to 6} |x - 6|$

Examples: Find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

1. $\lim_{x \to -3} (2x + 5) = -1$



Examples: Find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

1. $\lim_{x \to -3} (2x + 5) = -1$

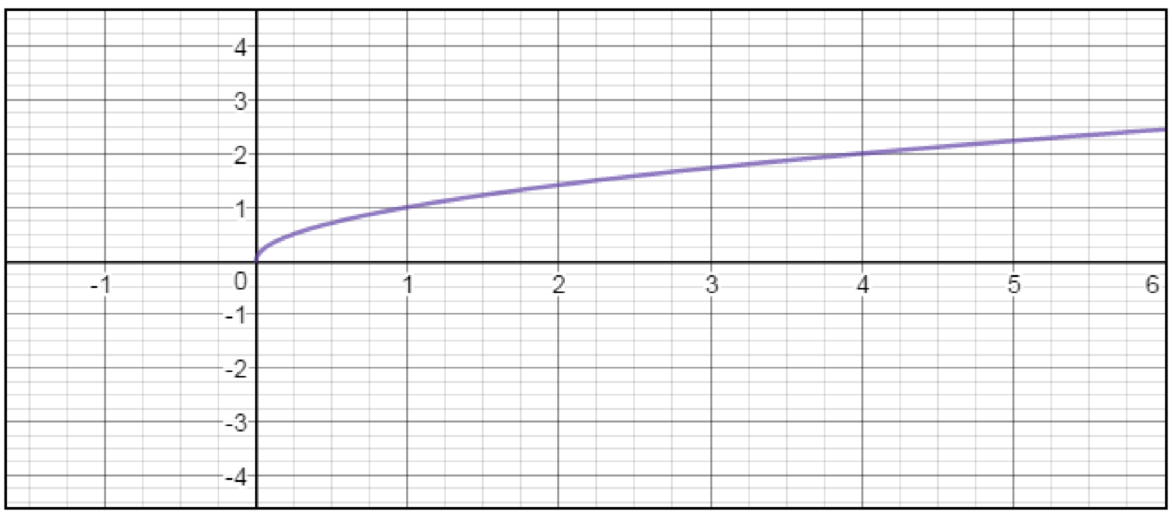
Keep your goal in mind: Transform $|f(x) - (-1)| < \varepsilon$ to $|x - (-3)| < \delta$.

$$|f(x) - (-1)| < \varepsilon$$
$$|2x + 5 - (-1)| < \varepsilon$$
$$|2x + 6| < \varepsilon$$
$$2|x + 3| < \varepsilon$$
$$|x + 3| < {}^{\varepsilon}/_{2}$$

We can get $f(x)$ as close to -1 as we want (ε), as long as x is within $\delta = {}^{\varepsilon}/_{2}$ of -3.

Examples: Find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

2. $\lim_{x \to 4} \sqrt{x} = 2$



Examples: Find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

2. $\lim\limits_{x \to 4} \sqrt{x} = 2$

$$|f(x) - L| < \varepsilon$$
$$\left|\sqrt{x} - 2\right| < \varepsilon$$

Multiply by the conjugate on both sides...

$$\left|\sqrt{x} + 2\right|\left|\sqrt{x} - 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Now to handle the right side of the inequality...

Examples: Find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

2. $\lim_{x \to 4} \sqrt{x} = 2$

$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

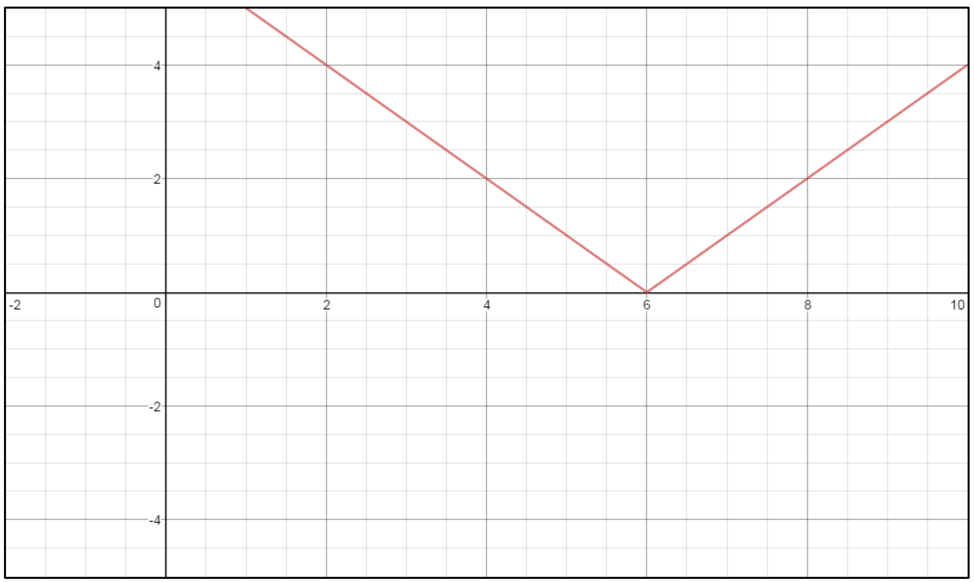
Near 4, $\left|\sqrt{x} + 2\right|$ is close to $\left|\sqrt{4} + 2\right| = 4$, so we can finish this off with

$$|x - 4| < \varepsilon(4)$$

That is, let $\delta = 4\varepsilon$

Examples: Find the limit L. Then use the $\delta - \varepsilon$ definition to prove that the limit is L.

3. $\lim_{x \to 6} |x - 6| = 0$



$$|f(x) - L| < \varepsilon$$

$$\big||x - 6| - 0\big| < \varepsilon$$

$$|x - 6| < \varepsilon$$

So let $\delta = \varepsilon$.

The End

- Thank you to Desmos.com and their free online graphing utility for the graphs.