Evaluating Limits Analytically

By Tuesday J. Johnson

Suggested Review Topics

- Algebra skills reviews suggested:
  - Evaluating functions
  - Rationalizing numerators and/or denominators

- Trigonometric skills reviews suggested:
  - Evaluating trigonometric functions
  - Basic quotient and reciprocal identities

Math 1411
Chapter 1: Limits and Their Properties

1.3 Evaluating Limits Analytically

Properties of Limits (1)

- Let b and c be real numbers and let n be a positive integer.
  1. $\lim_{x \to c} b = b$
     - The limit of a constant is the constant.
  2. $\lim_{x \to c} x = \mathrm{c}$
     - The limit of x is c.
  3. $\lim_{x \to c} x^n = c^n$

Properties of Limits (2)

- Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x \to c} f(x) = L \; and \; \lim_{x \to c} g(x) = K$$

1. Scalar Multiple:

$$\lim_{x \to c}[bf(x)] = bL$$

2. Sum or Difference:

$$\lim_{x \to c}[f(x) \pm g(x)] = L \pm K$$

3. Product:

$$\lim_{x \to c}[f(x)g(x)] = LK$$

Properties of Limits (2) continued

- Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x \to c} f(x) = L \; and \; \lim_{x \to c} g(x) = K$$

4. Quotient:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \qquad K \neq 0$$

5. Power:

$$\lim_{x \to c} [f(x)]^n = L^n$$

Limits of Polynomial and Rational Functions

- If p is a polynomial function and c is a real number, then $\lim_{x \to c} p(x) = p(c)$.
  - The limit of a polynomial can be found by evaluating the polynomial.
- If r is a rational function and given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then $\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$.
  - The limit of a rational function can be found by evaluating, assuming a nonzero denominator.

Limit of a Function Involving Radicals or Compositions

- Let n be a positive integer. The following limit is valid for all c if n is odd and is valid for $c > 0$ if n is even: $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$
  - Again, just evaluate if it is possible.
- If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} f(x) = f(L)$, then $\lim_{x \to c} f\big(g(x)\big) = f(\lim_{x \to c} g(x)) = f(L)$.

Limits of Trigonometric Functions

- Let c be a real number in the domain of the given trigonometric function.

1. $\lim\limits_{x \to c} \sin x = \sin c$
2. $\lim\limits_{x \to c} \cos x = \cos c$
3. $\lim\limits_{x \to c} \tan x = \tan c$
4. $\lim\limits_{x \to c} \cot x = \cot c$
5. $\lim\limits_{x \to c} \sec x = \sec c$
6. $\lim\limits_{x \to c} \csc x = \csc c$

That is a lot to remember…

- Don’t let yourself feel overwhelmed with the amount of material it seems you need to memorize.
- The first thing you need to know is which limits can be found by simply evaluating the function: In general, as long as c is in the domain of the function, you can evaluate the function to find the limit.
- If c is not in the domain of the function we must have some strategies in order to find the limit.

Theorem 1.7

- Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

- This says, essentially, that the limit of similar functions is the same.

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution.
2. If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. Common techniques involve factoring and canceling and rationalizing.
3. Apply Theorem 1.7 to conclude analytically that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c)$.
4. Use a graph or table to reinforce your conclusion.

Examples: Find the limits.

1. $\lim_{x\to 1}(-x^2+1)$
2. $\lim_{x\to 4}\sqrt[3]{x+4}$
3. $\lim_{x\to -3}\frac{2}{x+2}$
4. $\lim_{x\to \pi}\tan x$
5. $\lim_{x\to 5\pi/3}\cos x$
6. $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$
7. $\lim_{x\to 2}\frac{x^3-8}{x-2}$
8. $\lim_{x\to 1}\frac{3x}{x^2+2x}$
9. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

Examples: Find the limits.

1. $\lim_{x \to 1} (-x^2 + 1) = -(1)^2 + 1 = -1 + 1 = 0$
   – Evaluate a polynomial to find the limit. Don't overthink it!

2. $\lim_{x \to 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = \sqrt[3]{8} = 2$
   – Evaluate any radical with odd index to find the limit.

Examples: Find the limits.

3. $\lim_{x \to -3} \frac{2}{x+2} = \frac{2}{(-3)+2} = \frac{2}{-1} = -2$

- Evaluate since -3 is in the domain of the rational function.

4. $\lim_{x \to \pi} \tan x = \tan \pi = 0$

- Evaluate.

Examples: Find the limits.

5. $\lim\limits_{x \to 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$
   - Evaluate.

6. $\lim\limits_{x \to -1} \frac{2x^2 - x - 3}{x+1} = \lim\limits_{x \to -1} \frac{(2x-3)\cancel{(x+1)}}{\cancel{x+1}} =$
$\lim\limits_{x \to -1} (2x - 3) = 2(-1) - 3 = -5$
   - Since -1 is not in the domain, we factor and cancel to use Theorem 1.7.

Examples: Find the limits.

7. $\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{x-2}} =$
$\lim_{x\to 2}(x^2+2x+4) = (2)^2+2(2)+4 = 12$

8. $\lim_{x\to 1} \frac{3x}{x^2+2x} = \frac{3(1)}{(1)^2+2(1)} = \frac{3}{3} = 1$

– Remember, if the value c is in the domain, we can just evaluate to find the limit.

Examples: Find the limits.

9. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

– If we try to evaluate we get $0/0$. Anytime, and every time, we get $0/0$ we must do something to simplify.

factoring won't help so we rationalize the numerator:

$$\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} = \frac{\cancel{x-3}}{\cancel{(x-3)}(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2}$$

Examples: Find the limits.

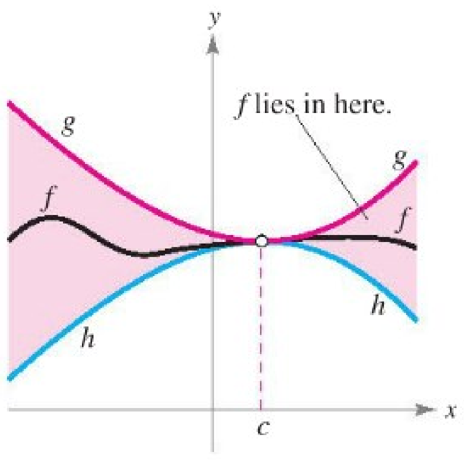
9. $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{4}$

- Study Tip: Always try to evaluate the limit by evaluating the given function. If you get a number, great. If you get $0/0$ there is ALWAYS something that can be done to find another function g in order to use Theorem 1.7.

The Squeeze (Sandwich) Theorem

- If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$ then $\lim_{x \to c} f(x)$ exists and is equal to L.

$h(x) \leq f(x) \leq g(x)$



The Squeeze Theorem

The Squeeze (Sandwich) Theorem

- Two special trigonometric limits come directly from an application of this theorem:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad and \quad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

❑ Study Tip: Write these down. They will come back several times.

Examples: Determine the limit of the trig function, if it exists.

1. $\lim\limits_{x \to 0} \dfrac{3(1-\cos x)}{x}$
2. $\lim\limits_{\theta \to 0} \dfrac{\cos\theta \tan\theta}{\theta}$
3. $\lim\limits_{\varphi \to \pi} \varphi \sec\varphi$

Examples: Determine the limit of the trig function, if it exists.

1. $\lim_{x\to 0} \frac{3(1-\cos x)}{x} = 3(\lim_{x\to 0} \frac{1-\cos x}{x}) = 3(0) = 0$

2. $\lim_{\theta\to 0} \frac{\cos\theta \tan\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

3. $\lim_{\varphi\to\pi} \varphi \sec\varphi = \lim_{\varphi\to\pi} \frac{\varphi}{\cos\varphi} = \frac{\pi}{\cos\pi} = \frac{\pi}{-1} = -\pi$

The End

http://www.larsoncalculus.com/calc10/content/proof-videos/chapter-1/section-3/proof-two-special-trigonometric-limits/