Continuity and One-Sided Limits

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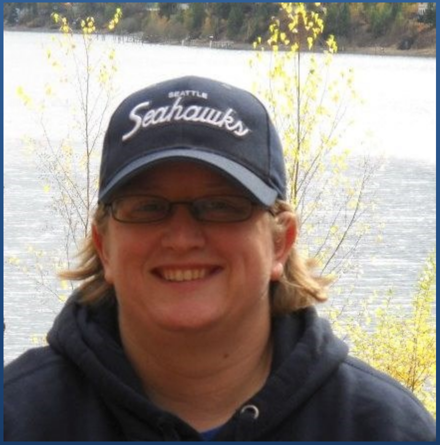


Suggested Review Topics

- Algebra skills reviews suggested:
  - Evaluating functions
  - Rationalizing numerators and/or denominators

- Trigonometric skills reviews suggested:
  - Evaluating trigonometric functions
  - Basic quotient and reciprocal identities

Math 1411
Chapter 1: Limits and Their Properties

1.4 Continuity and One-sided Limits

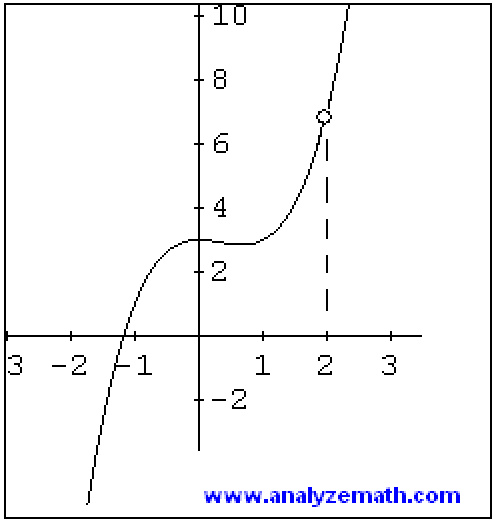
Continuity

- Definition – A function f is continuous at c if the following three conditions are met.
  1. $f(c)$ is defined
  2. $\lim_{x \to c} f(x)$ exists
  3. $\lim_{x \to c} f(x) = f(c)$

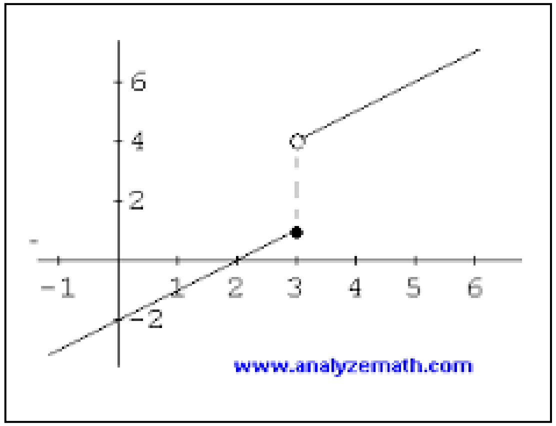
- A function is continuous on an open interval (a, b) if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

Examples: Ways a function can fail to be continuous.

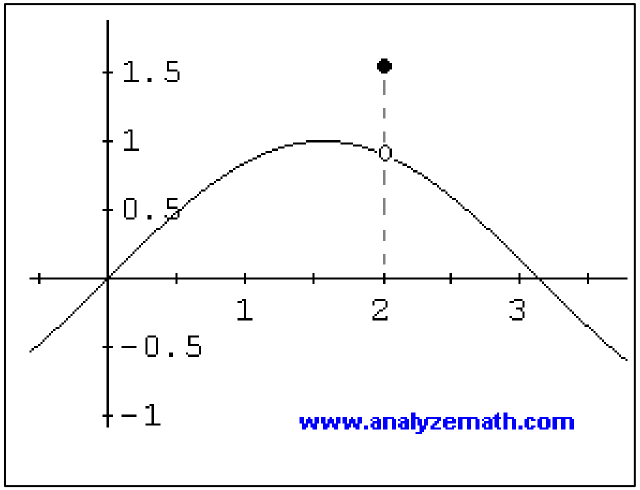
- The definition has three distinct parts, the function must be defined to be continuous. The function must have a limit to be continuous. AND those values must be equal.



1. Not Defined



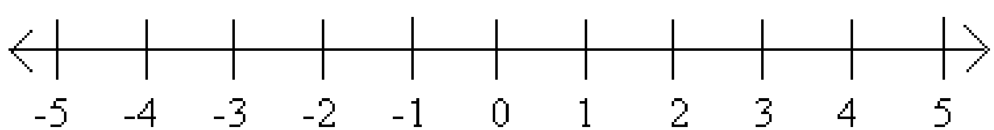
2. No limit



3. Limit does not equal function

One-sided Limits

- Frequently a function will have different behavior to the left and right of the value c under question. When this happens we look at one-sided limits.
- Consider a number line with negatives to the left and positives to the right:



One-sided Limits Defined

- That number line helps us define our one-sided limits as follows.

  1. From the right: $\lim_{x \to c^+} f(x) = L$
  2. From the left: $\lim_{x \to c^-} f(x) = L$

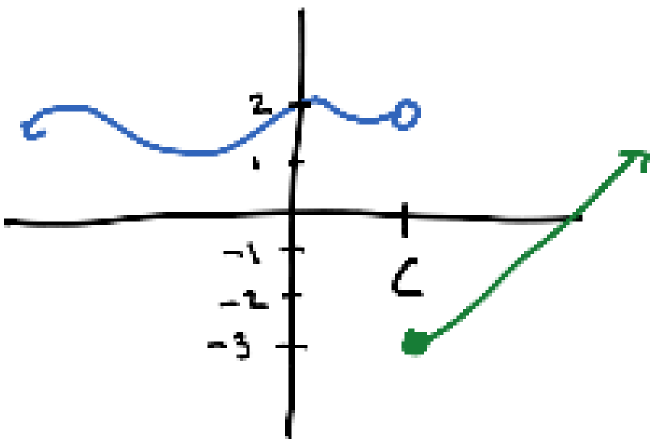
❑ Study Tip: From the right (side of the number line) is positive and from the left (side of the number line) is negative.

Existence of a Limit

- Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if
$$\lim_{x\to c^-} f(x) = L \quad and \quad \lim_{x\to c^+} f(x) = L$$

- This says that the limit exists overall if and only if the limits from each side exist AND are equal to each other.

One-sided Limits Graphically



As $x \to c^-$ (x approaches c from the left) we see that the graph gets close to the y-value of 2.

As $x \to c^+$ (x approaches c from the right) we see that the graph gets close to the y-value of -3.

Since $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, the limit does not exist. That is, the limit from the left does not equal the limit from the right so the overall limit does not exist.

Continuity on a Closed Interval

A function f is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and the limit from the right as x approaches a is equal to $f(a)$ and the limit from the left as x approaches b is equal to $f(b)$.

Examples: Find the limit, if it exists.

1. $\lim_{x \to 2^+} \frac{2-x}{x^2-4}$

2. $\lim_{x \to 10^+} \frac{|x-10|}{x-10}$

3. $\lim_{x \to \pi} \cot x$

4. $\lim_{x \to 4^+} (5\|x\| - 7)$

5. $\lim_{x \to 3^-} f(x)$ where $f(x) = \begin{cases} \frac{x+2}{2}, & x \leq 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases}$

Examples: Find the limit, if it exists.

1. $$\lim_{x\to 2^+} \frac{2-x}{x^2-4} = \lim_{x\to 2^+} \frac{-1\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2^+} \frac{-1}{x+2} = \frac{-1}{4}$$

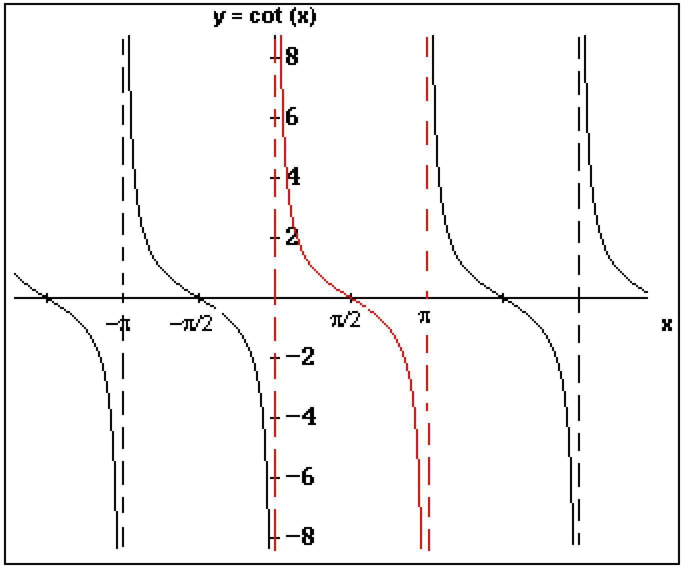
2. $$\lim_{x\to 10^-} \frac{|x-10|}{x-10} = -1.$$

   – Notice that the numerator and denominator are the same except possibly in sign. For $x \to 10^-$, the values to the left of 10 are smaller than 10 so $x - 10$ will be negative. This leads to our result.

Examples: Find the limit, if it exists.

3. $\lim_{x \to \pi} \cot x = \lim_{x \to \pi} \frac{\cos x}{\sin x} = \frac{-1}{0} = DNE$

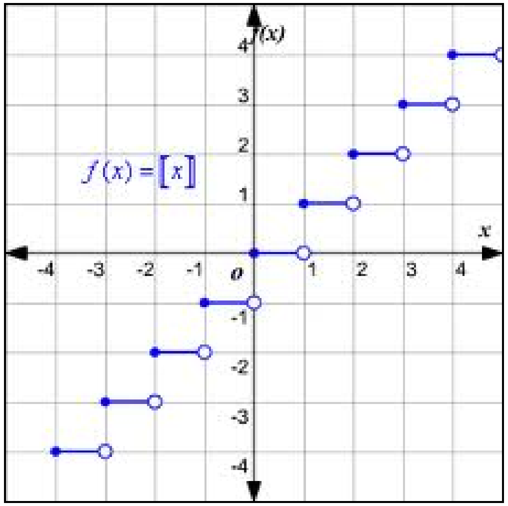
– We cannot divide by zero so the limit does not exist (DNE). We can see that the limit from the right is not equal to the limit from the left by considering the graph of cotangent and thinking of the location of asymptotes.



Examples: Find the limit, if it exists.

4. $\lim_{x \to 4^+} (5[\![x]\!] - 7) = 5(4) - 7 = 13$

- Recall that $[\![x]\!]$ represents the greatest integer function.
- For values close to 4, but slightly above (to the right), the greatest integer will yield 4.



Examples: Find the limit, if it exists.

5. $\lim_{x \to 3^-} f(x)$ where $f(x) = \begin{cases} \frac{x+2}{2}, & x \leq 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases}$

   – This is asking for the limit to the left of 3. In this case we would only consider the piece of the function that is defined to the left.

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x+2}{2} = \frac{3+2}{2} = \frac{5}{2}$$

Properties of Continuity

- If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.
  1. Scalar Multiple: bf
  2. Sum or Difference: $f \pm g$
  3. Product: fg
  4. Quotient: f/g, if $g(x) \neq 0$
- This means that most elementary functions are continuous on their domains. The key points to consider are values NOT in the domain.

Examples: Find the constants a and b such that the function is continuous on the entire real line.

1. $f(x) = \begin{cases} 2x^2, x \geq 1 \\ ax - 3, x < 1 \end{cases}$

2. $g(x) = \begin{cases} \frac{9 \sin x}{x}, \ x < 0 \\ a - 8x, \ x \geq 0 \end{cases}$

3. $h(x) = \begin{cases} 4, & x < -3 \\ ax - b, & -3 \leq x \leq 2 \\ -4, & x > 2 \end{cases}$

Examples: Find the constant a such that the function is continuous on the entire real line.

1. $f(x) = \begin{cases} 2x^2, x \geq 1 \\ ax - 3, x < 1 \end{cases}$

To be continuous, the limit from the left must be equal to the limit from the right AND be equal to the function value.

Examples: Find the constant a such that the function is continuous on the entire real line.

1. $f(x) = \begin{cases} 2x^2, x \geq 1 \\ ax - 3, x < 1 \end{cases}$

From the left:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax - 3) = a(1) - 3 = a - 3$$

From the right:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2) = 2(1)^2 = 2$$

Examples: Find the constant a such that the function is continuous on the entire real line.

1. $f(x) = \begin{cases} 2x^2, x \geq 1 \\ ax - 3, x < 1 \end{cases}$

From the left the limit is $a - 3$.

From the right the limit is 2.

For this function to be continuous it must be that a – $3 \ = \ 2$ so $a \ = \ 5$.

Examples: Find the constant a such that the function is continuous on the entire real line.

$$2.\ \ g(x) = \begin{cases} \dfrac{9\sin x}{x}, & x < 0 \\ a - 8x, & x \geq 0 \end{cases}$$

From the left:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \frac{9\sin x}{x} = 9\left(\lim_{x \to 0^-} \frac{\sin x}{x}\right) = 9(1) = 9$$

From the right:

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (a - 8x) = a$$

This means that it must be that $a = 9$.

Examples: Find the constant a and b such that the function is continuous on the entire real line.

$$3.\ h(x) = \begin{cases} 4, & x < -3 \\ ax - b, & -3 \le x \le 2 \\ -4, & x > 2 \end{cases}$$

This one is slightly different that the ones that came before. Our strategy will be to see what needs to happen to make the function continuous at the first cut point, then see what needs to happen at the second cut point. We will use this information to get a system of equations that will allow us to solve for the missing values.

$$3.\ h(x) = \begin{cases} 4, & x < -3 \\ ax - b, & -3 \le x \le 2 \\ -4, & x > 2 \end{cases}$$

At $x = -3$

- The limit from the right is $\lim_{x \to -3^+} (ax - b) = -3a - b$
- The limit from the left is $\lim_{x \to -3^-} (4) = 4.$
- To be continuous we must have $-3a - b = 4.$
- This doesn't allows us to solve for either a or b.

At $x = 2$

- The limit from the right is $\lim_{x \to 2^+} (-4) = -4.$
- The limit from the left is $\lim_{x \to 2^-} (ax - b) = 2a - b.$
- To be continuous it must be that $2a - b = -4.$
- Once again, by itself this does not allow us to solve for the missing values; but using the equations together we have a system.

Examples: Find the constants a and b such that the function is continuous on the entire real line.

$$3.\ \ h(x) = \begin{cases} 4, & x < -3 \\ ax - b, & -3 \leq x \leq 2 \\ -4, & x > 2 \end{cases}$$

Solve $\begin{cases} -3a - b = 4 \\ 2a - b = -4 \end{cases}$ to find that $a = -\frac{8}{5}$ and $b = \frac{4}{5}$.

Examples: Find the x-values (if any) at which *f* is not continuous. Which of the discontinuities are removable?

1. $f(x) = \frac{3}{x-2}$
2. $f(x) = x^2 - 2x + 1$
3. $f(x) = \frac{1}{x^2+1}$
4. $f(x) = \frac{x}{x^2-1}$
5. $f(x) = \frac{x-6}{x^2-36}$
6. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Examples: Find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

Fact: A discontinuity is removable if it can be factored out or otherwise dealt with.

1. $f(x) = \frac{3}{x-2}$

This function is not continuous at x = 2 (we only look at values not in the domain). This is not removable.

Examples: Find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

2. $f(x) = x^2 - 2x + 1$

This is a polynomial so everywhere continuous.

Examples: Find the x-values (if any) at which *f* is not continuous. Which of the discontinuities are removable?

3. $f(x) = \frac{1}{x^2+1}$

The domain consists of all real numbers so this function is continuous everywhere.

Examples: Find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

4. $f(x) = \frac{x}{x^2-1}$

Not continuous at $x = \pm 1$, not removable.

Examples: Find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

5. $f(x) = \frac{x-6}{x^2-36}$

We can simplify $f(x) = \frac{x-6}{x^2-36} = \frac{x-6}{(x-6)(x+6)}$. First, notice that there are discontinuities at both $x = 6$ and $x = -6$. However, the discontinuity at $x = 6$ IS removable as it can be factored out.

Examples: Find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

6. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

We must only check the cut value to see if each of these polynomial pieces meet up to make a continuous function. Why? Each piece is continuous on its given domain. Therefore the only possible discontinuity is at the cut value.

Examples: Find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

6. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1$$

Examples: Find the x-values (if any) at which *f* is not continuous. Which of the discontinuities are removable?

6. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Both the limit from the left and the limit from the right and the function value equal 1, therefore this function is continuous everywhere.

Facts and Theorems

- **Continuity of a Composite Function** – If *g* is continuous at *c* and *f* is continuous at *g(c),* then the composite function given by $(f \circ g)(x) = (f(g(x))$ is continuous at *c*.
- **Intermediate Value Theorem (IVT)** – If *f* is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number *c* in $[a, b]$ such that $f(c) = \mathrm{k}$.
- *Basic Interpretation* – While driving from home to work I start at 0 mph when I get in my car and go 75 mph on the highway. At some point in time I must have been going 55 mph.

Examples: Verify that the IVT applies to the indicated interval and find the value of *c* guaranteed by the theorem.

1. $f(x) = x^2 - 6x + 8, \;\; [0,3], \;\; f(c) = 0$

2. $f(x) = \frac{x^2+x}{x-1}, \;\; [\frac{5}{2}, 4], \;\; f(c) = 6$

Examples: Verify that the IVT applies to the indicated interval and find the value of *c* guaranteed by the theorem.

1. $f(x) = x^2 - 6x + 8, \ [0, 3], \ f(c) = 0$

Verify that the IVT applies:

a) f is a polynomial so is continuous

b) $f(a) = f(0) = 8$ and $f(b) = f(3) = -1$ so the values of the endpoints are not equal.

c) The value $k = 0$ is between -1 and 8.

Examples: Verify that the IVT applies to the indicated interval and find the value of *c* guaranteed by the theorem.

1. $f(x) = x^2 - 6x + 8, \ \ [0, 3], \ \ f(c) = 0$

Find the value c:

$$c^2 - 6c + 8 = 0$$
$$(c - 4)(c - 2) = 0$$
$$c = 4, c = 2$$

Our solution is $c = 2$ as it is the only result in the interval $[0,3]$.

Examples: Verify that the IVT applies to the indicated interval and find the value of *c* guaranteed by the theorem.

2. $f(x) = \frac{x^2+x}{x-1}$, $[\frac{5}{2}, 4]$, $f(c) = 6$

Verify:

a) f is continuous everywhere except at x = 1 which is not in the given interval.

b) $f\left(\frac{5}{2}\right) = \frac{35}{6}$, a little less than 6, and $f(4) = \frac{20}{3}$, a little more than 6.

c) 6 is between the two function values.

Examples: Verify that the IVT applies to the indicated interval and find the value of *c* guaranteed by the theorem.

2. $f(x) = \frac{x^2+x}{x-1},\ [\frac{5}{2}, 4],\ f(c) = 6$

Find the value c:

$$\frac{c^2 + c}{c - 1} = 6$$

Multiply both sides by the denominator to get

$$\cancel{(c-1)} \cdot \frac{c^2 + c}{\cancel{c - 1}} = 6(c - 1)$$

$$c^2 + c = 6c - 6$$

Examples: Verify that the IVT applies to the indicated interval and find the value of *c* guaranteed by the theorem.

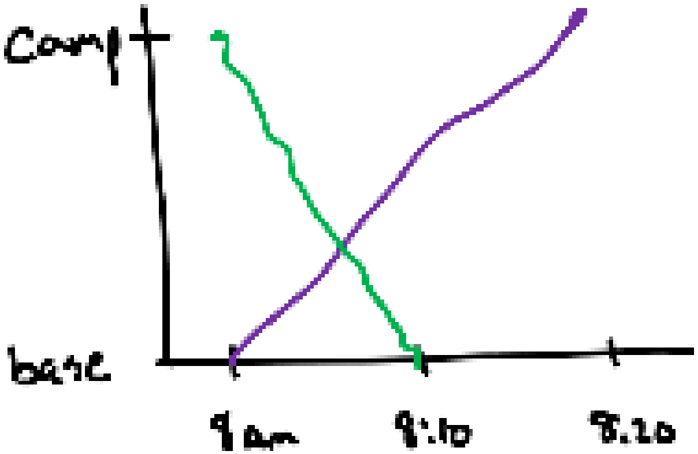
2. $f(x) = \frac{x^2+x}{x-1},\ [\frac{5}{2}, 4],\ \ f(c) = 6$

$$c^2 + c = 6c - 6$$
$$c^2 - 5c + 6 = 0$$
$$(c-3)(c-2) = 0$$
$$c = 3, c = 2$$

Our answer is c $=$ 3 as the other is not in the given interval.

Example

- Example: At 8:00 AM on Saturday a man begins running up the side of a mountain to his weekend campsite. On Sunday morning at 8:00 AM he runs back down the mountain. It take him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct.



Saturday

Sunday

Notice that in leaving at 8 AM both days, the graph shows his trail must intersect. This intersection is when he was at the same place at the same time both Saturday and Sunday.

The End