Infinite Limits

By Tuesday J. Johnson

Suggested Review Topics

- Algebra skills reviews suggested:
  - Evaluating functions
  - Graphing functions
  - Working with inequalities
  - Working with absolute values
- Trigonometric skills reviews suggested:
  - Graphing trigonometric functions
  - Evaluating trigonometric functions

Math 1411
Chapter 1: Limits and Their Properties

1.5 Infinite Limits

Definition: **Infinite Limits**

- Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement $\lim_{x \to c} f(x) = \infty$ means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.
- Similarly, the statement $\lim_{x \to c} f(x) = -\infty$ means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.

- To define the infinite limit from the left, replace $0 < |x - c| < \delta$ by $c - \delta < x < c$.
- To define the infinite limit from the right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.
- **Note 1:** Having a limit equal to infinity does NOT mean the limit exists. In fact, it means the limit is unbounded and therefore does not exist.
- **Note 2:** In WebAssign, if the limit is ∞ you should enter that for your answer and not DNE. There does not seem to be a great consistency in this however.

Vertical Asymptotes

- Definition – If $f(x)$ approaches infinity (positive or negative) as *x* approaches *c* from the right or the left, then the line $x = c$ is a vertical asymptote of the graph of $f(x)$.
- Theorem – Let *f* and *g* be continuous on an open interval containing *c*. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing *c* such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by $h(x) = \frac{f(x)}{g(x)}$ has a vertical asymptote at $x = c$.

Examples: Find the vertical asymptote, if any.

1. $f(x) = \frac{-4x}{x^2+4}$

- The numerator is zero when x = 0
- The denominator is never zero for real values of x.

- Therefore there is no vertical asymptote.

Examples: Find the vertical asymptote, if any.

2. $h(s) = \frac{2s-3}{s^2-25}$

- The numerator is zero at $s = \frac{3}{2}$.
- The denominator is zero at $s = \pm 5$.
- Therefore the vertical asymptotes are $s = 5$ and $s = -5$.

❖ Note: Vertical asymptotes are ALWAYS equations of lines, never just values.

Examples: Find the vertical asymptote, if any.

3. $g(x) = \frac{x^3+1}{x+1} = \frac{(x+1)(x^2+x+1)}{x+1}$

- Both numerator and denominator are zero at $x = -1$. This is NOT a vertical asymptote.
- This situation will result in a removable discontinuity.

Examples: Find the vertical asymptote, if any.

4. $h(t) = \frac{t^2-2t}{t^4-16} = \frac{t(t-2)}{(t-2)(t+2)(t^2+4)}$

- The numerator is zero at $t = 0$ and $t = 2$.
- The denominator is zero at $t = 2$ and $t = -2$.
- This tells us that $t = 2$ is NOT a vertical asymptote as it is a removable discontinuity.
- However, $t = -2$ is a vertical asymptote.

Examples: Find the vertical asymptote(s), if any.

5. $f(x) = \sec(\pi x) = \dfrac{1}{\cos(\pi x)}$

- The numerator is never zero.
- The denominator is zero when $\pi x = \frac{\pi}{2} + \pi n$, where n is an arbitrary integer.
- This means that $f(x) = \sec(\pi x)$ has vertical asymptotes at $x = ½ + n$, for all integers n.

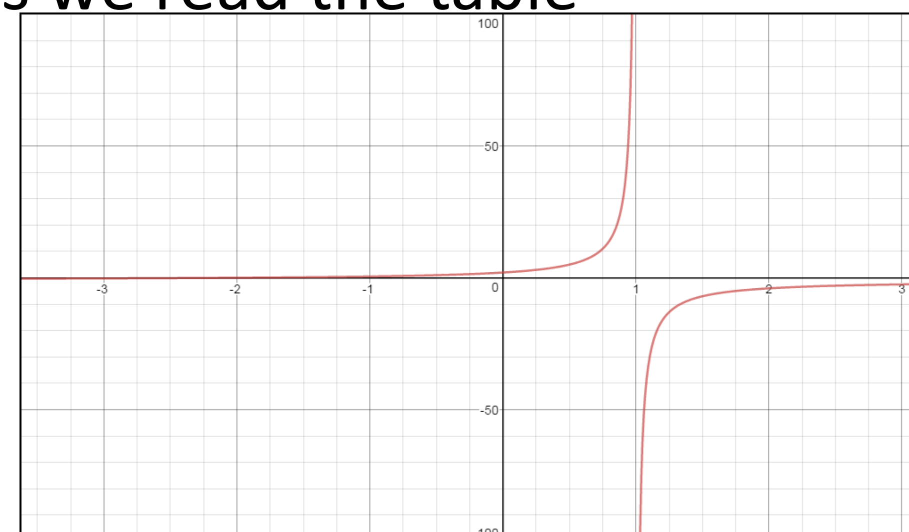
Examples: Find the limit, if it exists.

1. $\lim\limits_{x \to 1^+} \dfrac{2+x}{1-x}$

Tables are great if you don't have access to a graph:

x	1	1.001	1.01	1.1
$f(x)$	?	-3001	-301	-31

x gets large and negative as we read the table right to left getting closer to 1 (from the right).



This means that

$$\lim_{x \to 1^+} \frac{2+x}{1-x} = -\infty.$$

Examples: Find the limit, if it exists.

2. $\lim_{x \to 4^+} \frac{x^2}{x^2+16}$

We should always evaluate to find a limit if the domain allows. In this case that is the best method:

$$\lim_{x \to 4^+} \frac{x^2}{x^2 + 16} = \frac{16}{16 + 16} = \frac{1}{2}$$

We could have used a table or graph, but when possible the best thing to do is evaluate.

Examples: Find the limit, if it exists.

3.

$$\lim_{x \to (\frac{\pi}{2})^+} \left(\frac{-2}{\cos x}\right) = \lim_{x \to (\frac{\pi}{2})^+} (-2\sec x)$$

- If we try to evaluate we get $\frac{-2}{0}$, but that doesn't necessarily mean our solution is negative infinity.
- If we try a table we realize we are in radians, what values does that suggest to get "close to" $\frac{\pi}{2}$ from the right?
- Let's take a look at this graph!

Examples: Find the limit, if it exists.

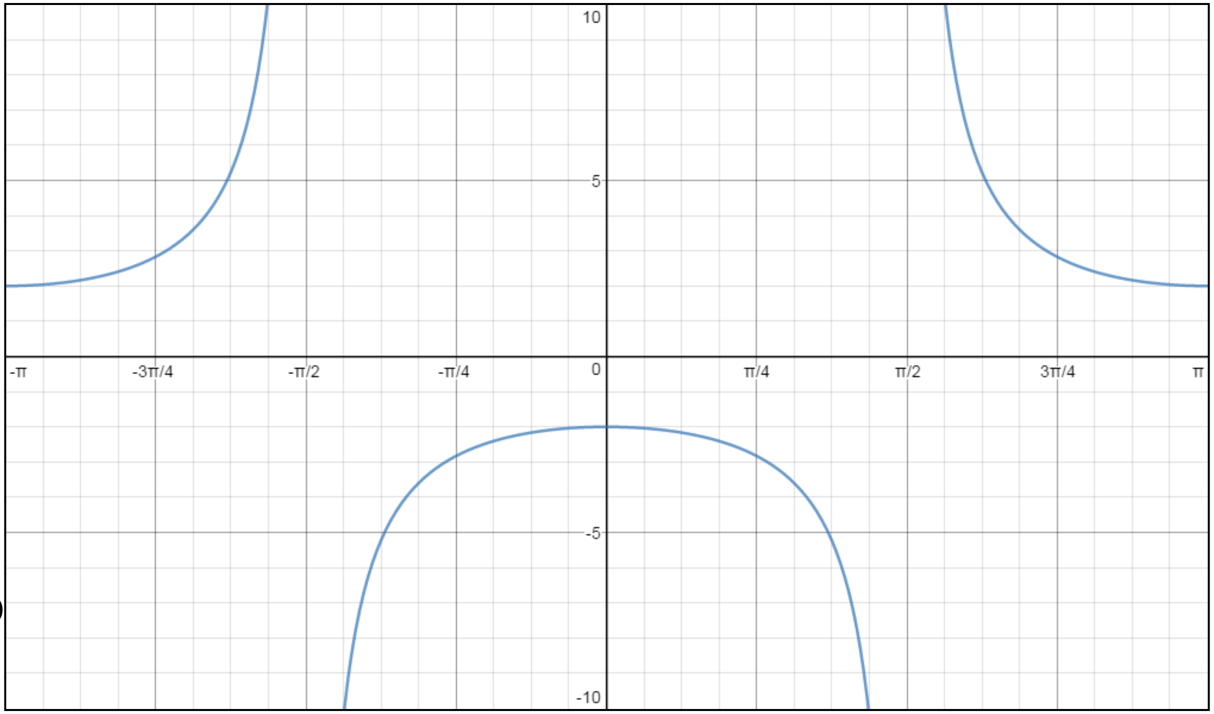
3.

$$\lim_{x \to (\frac{\pi}{2})^+} \left(\frac{-2}{\cos x}\right) = \lim_{x \to (\frac{\pi}{2})^+} (-2 \sec x)$$

As $x \to \frac{\pi}{2}^+$, the graph of $f(x)$ approaches $+\infty$.

Therefore,

$$\lim_{x \to (\frac{\pi}{2})^+} \left(\frac{-2}{\cos x}\right) = \infty$$



Properties of Infinite Limits

Let *c* and *L* be real numbers and let *f* and *g* be functions such that $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$.

1. Sum or difference: $\lim_{x \to c}[f(x) \pm g(x)] = \infty$.

2. Product: $\lim_{x \to c}[f(x)g(x)] = \infty, \quad L > 0$ and

   $\lim_{x \to c}[f(x)g(x)] = -\infty, \quad L < 0$.

3. Quotient: $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of *f(x)* as *x* approaches *c* is negative infinity.

The End

- Thank you to Desmos.com and their free online graphing utility for the graphs.