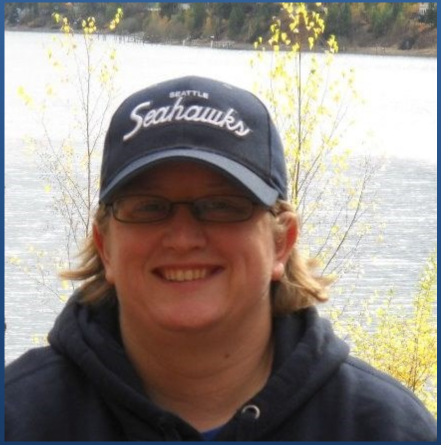
The Derivative and Tangent Line Problem

By Tuesday J. Johnson



Suggested Review Topics

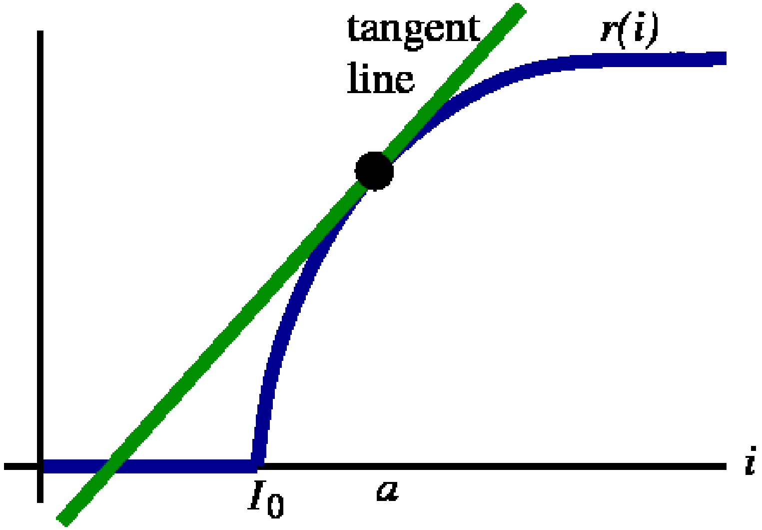
- Algebra skills reviews suggested:
  - Evaluating functions
  - Difference Quotients
  - Writing linear equations using point-slope form

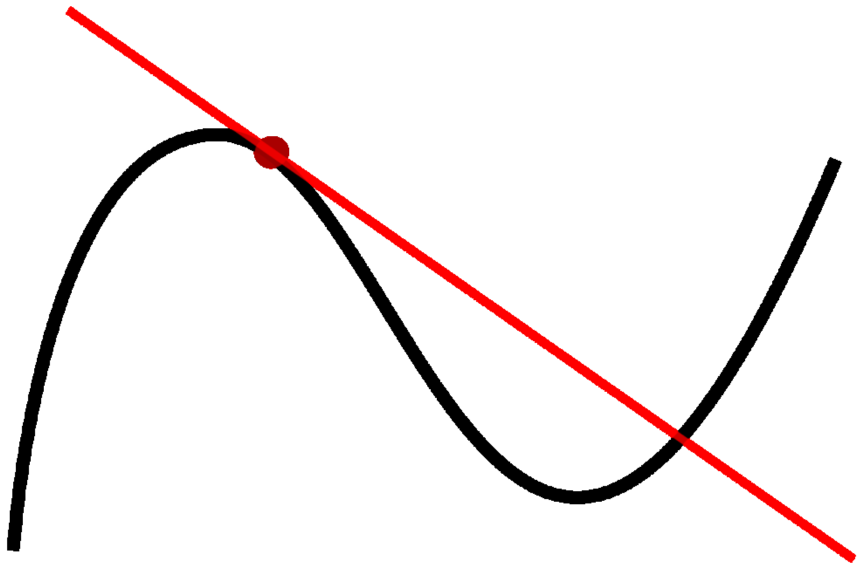
- Trigonometric skills reviews suggested:
  - None

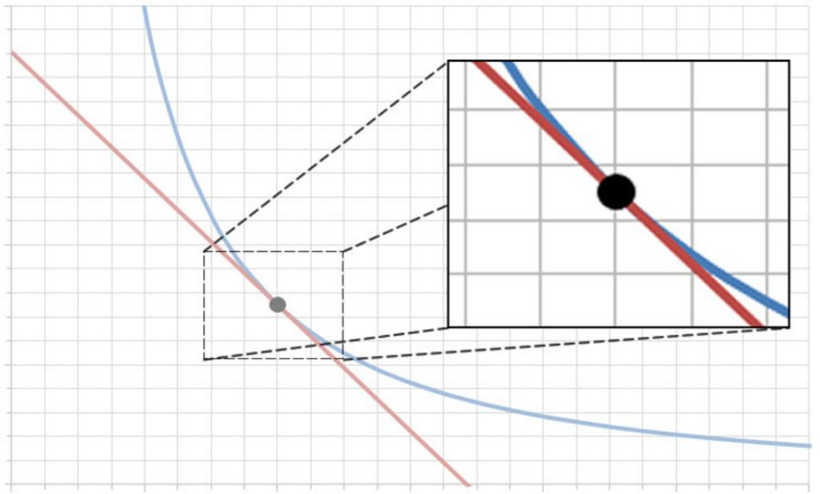
Math 1411
Differentiation

The Derivative and the Tangent Line Problem

What is a tangent line?







Difference Quotient

- The difference quotient from pre-calculus is a generalized form of the slope formula and is given by $\frac{f(x+h)-f(x)}{h}$.
- This difference quotient forms the basis for what is essentially half of calculus: the derivative.

Definition

- **Tangent Line with Slope m** – If f is defined on an open interval containing *c*, and if the limit $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = m$ exists, then the line passing through $(c, f(c))$ with slope *m* is the tangent line to the graph of *f* at the point $(c, f(c))$.
- **Fact:** The slope of the tangent line is also called the slope of the graph.

Examples: Find the slope of the tangent line to the graph of the function at the given point.

1. $f(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

To find the slope of the tangent line we will take the difference quotient in steps:

$$f(-2+h) = \frac{3}{2}(-2+h) + 1 = -3 + \frac{3}{2}h + 1$$

$$= -2 + \frac{3}{2}h$$

$$f(-2) = \frac{3}{2}(-2) + 1 = -3 + 1 = -2$$

Examples: Find the slope of the tangent line to the graph of the function at the given point.

1. $f(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

Now we use the difference quotient to find the slope: $\lim\limits_{h\to 0} \frac{(-2+\frac{3}{2}h)-(-2)}{h} = \lim\limits_{h\to 0} \frac{\frac{3}{2}h}{h} = \lim\limits_{h\to 0} \frac{3}{2} = \frac{3}{2}$.
The slope of the tangent line to the graph of our function at $c = -2$ is $3/2$.

- Note: The slope of any linear function is also going to be the slope of the tangent line.

Examples: Find the slope of the tangent line to the graph of the function at the given point.

2. $g(x) = 6 - x^2, \quad (1, 5)$

Here $c = 1$:

$$g(1+h) = 6 - (1+h)^2 = 6 - (1 + 2h + h^2)$$
$$= 6 - 1 - 2h - h^2 = 5 - 2h - h^2$$

$g(1) = 5$ and so the slope of the tangent line is

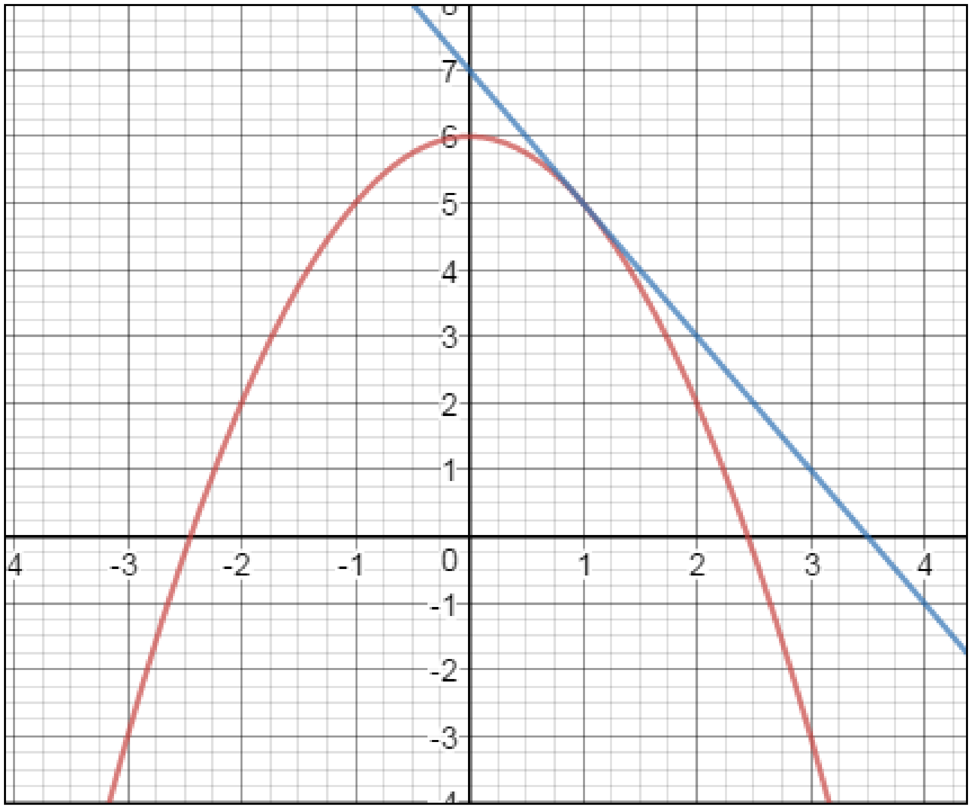
$$\lim_{h \to 0} \frac{(5-2h-h^2)-(5)}{h} = \lim_{h \to 0} \frac{-2h-h^2}{h} =$$

$$\lim_{h \to 0} \frac{\cancel{h}(-2-h)}{\cancel{h}} = \lim_{h \to 0}(-2-h) = -2$$

Examples: Find the slope of the tangent line to the graph of the function at the given point.

2. $g(x) = 6 - x^2, \quad (1, 5)$

The slope of the tangent line is $m = -2$



Examples: Find the slope of the tangent line to the graph of the function at the given point.

3. $r(t) = t^2 + 3, \quad (-2,7)$

Using $c = -2$ we find that $r(-2+h) = 7 - 4h + h^2$ and $r(-2) = 7$. Using this in the difference quotient we have

$$\lim_{h\to 0} \frac{(7 - 4h + h^2) - 7}{h} =$$

$$\lim_{h\to 0} \frac{-4h + h^2}{h} = \lim_{h\to 0}(-4 + h) = -4$$

Definition

- **The Derivative of a Function** – The derivative of *f* at *x* is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. For all *x* for which this limit exists, *f'* is a function of *x*.

- **Notation** – The following are equivalent:

$$f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y]$$

Examples: Find the derivative by the limit process.

1. $f(x) = 3x + 2$

Step 1: $f(x+h) = 3(x+h) + 2 = 3x + 3h + 2$

Step 2: $f(x) = 3x + 2$

Step 3: $f(x+h) - f(x) = (3x + 3h + 2) - (3x + 2) = 3h$

Step 4: $\lim\limits_{h \to 0} \frac{3h}{h} = \lim\limits_{h \to 0} 3 = 3$

That is, $f'(x) = 3$.

Examples: Find the derivative by the limit process.

2. $g(x) = 2 - x^2$

$$\begin{aligned} g(x+h) &= 2 - (x+h)^2 \\ &= 2 - (x^2 + 2xh + h^2) \\ &= 2 - x^2 - 2xh - h^2 \end{aligned}$$

$$g(x+h) - g(x) = -2xh - h^2$$

Examples: Find the derivative by the limit process.

2. $g(x) = 2 - x^2$

$$g(x+h) - g(x) = -2xh - h^2$$

$$g'(x) = \lim_{h \to 0} \frac{-2xh - h^2}{h} = \lim_{h \to 0}(-2x - h)$$
$$= -2x$$

Examples: Find the derivative by the limit process.

3. $f(x) = \frac{4}{\sqrt{x}}$

First, $f(x+h) = \frac{4}{\sqrt{x+h}}$. That seemed easy right? Maybe too easy? Now subtract to get

$$f(x+h) - f(x) = \frac{4}{\sqrt{x+h}} - \frac{4}{\sqrt{x}}$$

This is going to require algebra.

Examples: Find the derivative by the limit process.

3. $f(x) = \frac{4}{\sqrt{x}}$

$$\frac{4}{\sqrt{x+h}} - \frac{4}{\sqrt{x}} = \frac{4}{\sqrt{x+h}}\frac{\sqrt{x}}{\sqrt{x}} - \frac{4}{\sqrt{x}}\frac{\sqrt{x+h}}{\sqrt{x+h}}$$

$$= \frac{4\sqrt{x}}{\sqrt{x+h}\sqrt{x}} - \frac{4\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} = \frac{4\sqrt{x} - 4\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}$$

Examples: Find the derivative by the limit process.

3. $f(x) = \frac{4}{\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{4\sqrt{x} - 4\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h} = \frac{0}{0}$$

A common denominator wasn't enough, let's rationalize the numerator.

Examples: Find the derivative by the limit process.

3. $f(x) = \frac{4}{\sqrt{x}}$

$$= \frac{4\sqrt{x} - 4\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \frac{4\sqrt{x} + 4\sqrt{x+h}}{4\sqrt{x} + 4\sqrt{x+h}}$$

$$= \frac{16x - 16(x+h)}{(\sqrt{x+h}\sqrt{x})(4\sqrt{x} + 4\sqrt{x+h})}$$

$$= \frac{-16h}{(\sqrt{x+h}\sqrt{x})(4\sqrt{x} + 4\sqrt{x+h})}$$

Examples: Find the derivative by the limit process.

3. $f(x) = \frac{4}{\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{-16h}{(\sqrt{x+h}\sqrt{x})(4\sqrt{x} + 4\sqrt{x+h})}}{h}$$

$$= \lim_{h \to 0} \frac{-16}{(\sqrt{x+h}\sqrt{x})(4\sqrt{x} + 4\sqrt{x+h})}$$

$$= \frac{-16}{(\sqrt{x}\sqrt{x})(4\sqrt{x} + 4\sqrt{x})} = \frac{-16}{x(8\sqrt{x})} = \frac{-2}{x\sqrt{x}}$$

Examples: Find an equation of the tangent line to the graph of f at the given point.

Equation of tangent line: the equation of a line has form $y = mx + b$ so we need to find m and b. The best way to do this in calculus is to remember that $m = f'(x)$ and use the point-slope form of the equation of a line:

$$y - f(c) = f'(c)(x - c)$$

Our strategy is to find f'(c) and then use the point-slope form of the line to find the equation of the tangent line.

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

1. $f(x) = x^2 + 3x + 4$ at $(-2,2)$

First, we find the derivative:

$$f(x+h) = (x+h)^2 + 3(x+h) + 4$$
$$= x^2 + 2xh + h^2 + 3x + 3h + 4$$

$$f(x+h) - f(x) = 2xh + h^2 + 3h$$

$$\frac{f(x+h) - f(x)}{h} = 2x + h + 3$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

1. $f(x) = x^2 + 3x + 4$ at $(-2,2)$

$$\frac{f(x+h) - f(x)}{h} = 2x + h + 3$$
$$f'(x) = \lim_{h\to 0}(2x + h + 3) = 2x + 3$$

So $f'(-2) = 2(-2) + 3 = -4 + 3 = -1$

We now have everything we need in order to find the equation of the tangent line.

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

1. $f(x) = x^2 + 3x + 4$ at $(-2,2)$

With $f'(-2) = -1$, $c = -2$ and $f(c) = 2$ we can find the equation of the tangent line to be

$$y - 2 = -1\big(x - (-2)\big)$$
$$y - 2 = -1(x + 2)$$
$$y - 2 = -x - 2$$
$$y = -x$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

2. $f(x) = \sqrt{x-1}$ at $(5,2)$

We can find the derivative at a point, rather than finding the derivative function and then evaluating:

$$f(5+h) = \sqrt{5+h-1} = \sqrt{4+h}$$
$$f(5) = \sqrt{5-1} = \sqrt{4} = 2$$

$$f(5+h) - f(5) = \sqrt{4+h} - 2$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

2. $f(x) = \sqrt{x-1}$ at $(5,2)$

$$f(5+h) - f(5) = \sqrt{4+h} - 2$$

$$\frac{f(5+h) - f(5)}{h} = \frac{\sqrt{4+h} - 2}{h}$$

If we try to let h approach 0 we will get 0/0 so…

$$\left(\frac{\sqrt{4+h} - 2}{h}\right)\left(\frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}\right) = \frac{1}{\sqrt{4+h} + 2}$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

2. $f(x) = \sqrt{x-1}$ at $(5,2)$

$$f'(5) = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

The equation of the tangent line is therefore:

$$y - 2 = \frac{1}{4}(x-5)$$

$$y - 2 = \frac{1}{4}x - \frac{5}{4}$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

3. $f(x) = \frac{1}{x+1}$ at $(0,1)$

$$f(0+h) = \frac{1}{h+1} \text{ and } f(0) = 1$$

$$f(0+h) - f(0) = \frac{1}{h+1} - 1 = \frac{1}{h+1} - \frac{h+1}{h+1}$$

$$= \frac{-h}{h+1}$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

3. $f(x) = \frac{1}{x+1}$ at $(0,1)$

$$f'(0) = \lim_{h\to 0} \frac{\frac{-h}{h+1}}{h} = \lim_{h\to 0} \frac{-1}{h+1} = -1$$

$$y - 1 = -1(x - 0)$$
$$y = -x + 1$$

Example: Find an equation of a line that is tangent to $f(x) = x^3 + 2$ and is parallel to $3x - y - 4 = 0$.

- This problem has three main parts. First, parallel lines have equal slopes. Let's find the slope of the given line by solving for y.

$$3x - 4 = y$$

The given slope is $m = 3$.

- Next, we find all points on the graph of f that have a slope of the tangent line (derivative) equal to 3.

Example: Find an equation of the line that is tangent to $f(x) = x^3 + 2$ and is parallel to $3x - y - 4 = 0$.

$$f(x+h) = (x+h)^3 + 2$$
$$= x^3 + 3x^2h + 3xh^2 + h^3 + 2$$

$$f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$$

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2$$

Example: Find an equation of the line that is tangent to $f(x) = x^3 + 2$ and is parallel to $3x - y - 4 = 0$.

$$f'(x) = \lim_{h \to 0}(3x^2 + 3xh + h^2) = 3x^2$$

If $f'(x) = 3x^2$, then $f'(x) = 3$ when $3x^2 = 3$ which is when $x = \pm 1$. This gives us TWO tangent lines. One through $(-1,1)$ and one through $(1,3)$.

Example: Find an equation of the line that is tangent to $f(x) = x^3 + 2$ and is parallel to $3x - y - 4 = 0$.

- Our third, and last, step is to find the equations of both of these lines. Recall that the slope of each is 3.

One through $(-1,1)$: $y - 1 = 3(x + 1)$ which gives $y = 3x + 4$.

One through $(1,3)$: $y - 3 = 3(x - 1)$ which gives $y = 3x$.

Definition

- **Alternate Definition of the Derivative** – The derivative of f at c is $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ provided this limit exists.
- Notice that this quotient is just the formula for the slope of a line between two points and the limit is what makes it work for nonlinear function.

Examples: Use the alternative form of the derivative to find the derivative at $x = c$, if it exists.

1. $f(x) = x(x-1)$ when $c = 1$

$$f'(1) = \lim_{x \to 1} \frac{x(x-1) - 1(1-1)}{x-1}$$
$$= \lim_{x \to 1} \frac{x(x-1)}{x-1}$$
$$= \lim_{x \to 1} x = 1$$

That is, $f'(1) = 1$.

Examples: Use the alternative form of the derivative to find the derivative at $x = c$, if it exists.

2. $f(x) = \frac{2}{x}$ at $c = 5$

$$f'(5) = \lim_{x \to 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5}$$

Let’s take this a step at a time. First we will simplify the difference in the numerator, then we will simplify the overall fraction.

Examples: Use the alternative form of the derivative to find the derivative at $x = c$, if it exists.

2. $f(x) = \frac{2}{x}$ at $c = 5$

$$\frac{2}{x} - \frac{2}{5} = \frac{2}{x}\frac{5}{5} - \frac{2}{5}\frac{x}{x} = \frac{10}{5x} - \frac{2x}{5x} = \frac{10 - 2x}{5x}$$

$$\frac{\frac{10 - 2x}{5x}}{x - 5} = \frac{10 - 2x}{5x} \cdot \frac{1}{x - 5} = \frac{-2(x - 5)}{5x}\frac{1}{x - 5} = \frac{-2}{5x}$$

Examples: Use the alternative form of the derivative to find the derivative at $x = c$, if it exists.

2. $f(x) = \frac{2}{x}$ at $c = 5$

$$f'(5) = \lim_{x \to 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5} = \lim_{x \to 5} \frac{-2}{5x} = \frac{-2}{5(5)} = \frac{-2}{25}$$

The End

- Thank you to Desmos.com and their free online graphing utility for the graphs.