The Chain Rule

By Tuesday J. Johnson

Suggested Review Topics

- Algebra skills reviews suggested:
  - Composition of functions
- Trigonometric skills reviews suggested:
  - None

Calculus
Differentiation

The Chain Rule

Background Knowledge

- In order to completely understand the chain rule, you must understand composition of functions. If you need a review of this topic, please see http://www.math.utep.edu/Faculty/tuesdayj/math1508/1508Ch1Sec8.pdf or http://www.math.utep.edu/Faculty/tuesdayj/math120/120Ch3Sec3.pdf.
- Composition of functions is evaluating one function with another and is frequently written $y = f(g(x))$. I will refer to f as the "outside" function and g as the "inside" function.

The Chain Rule

- If $y = f(u)$ is a differentiable function of *u* and $u = g(x)$ is a differentiable function of *x*, then $y = f(g(x))$ is a differentiable function of *x* and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, or, equivalently,
$$\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)g'(x)$$
- In words, the derivative of a composition is the derivative of the outside, leave the inside alone, multiplied by the derivative of the inside.

The General Power Rule

- If $y = [u(x)]^n$, where *u* is a differentiable function of *x* and *n* is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

Examples: Find the derivative of the function.

1. $y = 2(6 - x^2)^5$

Inside is $u = 6 - x^2$ with $u' = -2x$

Outside is $y = 2u^5$ with $y' = 10u^4$

Putting these together we have

$$y' = 10u^4 \cdot (-2x)$$
$$y' = 10(6 - x^2)^4(-2x)$$
$$y' = -20x(6 - x^2)^4$$

Examples: Find the derivative of the function.

2. $g(x) = \sqrt{x^2 - 2x + 1}$

Inside is $u = x^2 - 2x + 1$with $u' = 2x - 2$

Outside is $y = \sqrt{u}$ with $y' = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$

Putting these together we have

$$g'(x) = \frac{1}{2\sqrt{u}} \cdot (2x - 2)$$

$$g'(x) = \frac{2x - 2}{2\sqrt{x^2 - 2x + 1}} = \frac{x - 1}{\sqrt{x^2 - 2x + 1}}$$

Examples: Find the derivative of the function.

3. $y = -\frac{5}{(t+3)^3}$

Inside is $u = t + 3$ with $u' = 1$

Outside is $y = -\frac{5}{u^3} = -5u^{-3}$ with $y' = 15u^{-4}$

$$y' = 15u^{-4}(1)$$

$$y' = \frac{15}{u^4} = \frac{15}{(t+3)^4}$$

Examples: Find the derivative of the function.

4. $y = \frac{x}{\sqrt{x^4+4}}$

The overriding operation here is the quotient rule:

$$y' = \frac{\sqrt{x^4+4}(1) - x\frac{d}{dx}\sqrt{x^4+4}}{\sqrt{x^4+4}^2}$$

But to find the derivative of the radical we need to use the chain rule.

Examples: Find the derivative of the function.

4. $y = \frac{x}{\sqrt{x^4+4}}$

Focusing on the denominator:

Inside is $u = x^4 + 4$ with $u' = 4x^3$

Outside is $w = \sqrt{u}$ with $w' = \frac{1}{2\sqrt{u}}$

Now the derivative of the denominator is

$$w' = \frac{1}{2\sqrt{u}}(4x^3) = \frac{2x^3}{\sqrt{x^4 + 4}}$$

Examples: Find the derivative of the function.

4. $y = \frac{x}{\sqrt{x^4+4}}$

Substituting what we now know:

$$y' = \frac{\sqrt{x^4+4} - x\dfrac{2x^3}{\sqrt{x^4+4}}}{x^4+4}$$

We should simplify, but we will leave it here anyway.

Examples: Find the derivative of the function.

5. $h(t) = (\frac{t^2}{t^3+2})^2$

Inside is $u = \frac{t^2}{t^3+2}$ with $u' = \frac{(t^3+2)(2t)-t^2(3t^2)}{(t^3+2)^2}$

Outside is $y = u^2$ with $y' = 2u$

$$h'(t) = 2u \cdot \frac{(t^3+2)(2t) - t^2(3t^2)}{(t^3+2)^2}$$

$$h'(t) = 2(\frac{t^2}{t^3+2})\frac{(2t^4+4t) - \ (3t^4)}{(t^3+2)^2}$$

$$h'(t) = \frac{2t^2(4t - t^4)}{(t^3+2)^3}$$

Examples: Find the derivative of the function.

6. $f(x) = (2 + (x^2 + 1)^4)^3$

This nested chain rule will take patience, but if you write out what we do at each step it becomes easier:

$$f'(x) = 3(2 + (x^2 + 1)^4)^2 \cdot \frac{d}{dx}(2 + (x^2 + 1)^4)$$

$$f'(x) = 3(2 + (x^2 + 1)^4)^2 \cdot (0 + 4(x^2 + 1)^3) \cdot \frac{d}{dx}(x^2 + 1)$$

$$f'(x) = 3(2 + (x^2 + 1)^4)^2 \cdot (4(x^2 + 1)^3) \cdot 2x$$

Examples: Find the derivative of the function.

$7.\ y = \sin \pi x = \sin(\pi x)$

The parentheses aren't always written, but you should always be aware of them.

$$y' = \cos(\pi x) \cdot \pi = \pi \cos(\pi x)$$

Examples: Find the derivative of the function.

8. $y = \cos(1-2x)^2$

Let $v = 1 - 2x$, $u = v^2$, and $y = \cos u$. Then

$$y' = -\sin u \frac{du}{dv}$$

$$y' = -\sin(v)^2 \cdot (2v)\frac{dv}{dx}$$

$$y' = -\sin(1-2x)^2 \cdot \big(2(1-2x)\big)(-2)$$

$$y' = 4(1-2x)\sin(1-2x)^2$$

Examples: Find the derivative of the function.

9. $y = 3x - 5\cos(\pi x)^2$

You try it first:

Examples: Find the derivative of the function.

9. $y = 3x - 5\cos(\pi x)^2$

$$y' = 3 - 5(-\sin(\pi x)^2)\frac{d}{dx}(\pi x)^2$$

$$y' = 3 - 5(-\sin(\pi x)^2)\,(2\pi x)\frac{d}{dx}(\pi x)$$

$$y' = 3 - 5(-\sin(\pi x)^2)\,(2\pi x)(\pi)$$

$$y' = 3 + 10\pi^2 x\sin(\pi x)^2$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

1. $y = (4x^3 + 3)^2$ at $(-1,1)$

An equation of a tangent line needs a slope and a point. We have the point, the slope will be found by evaluating the derivative. Let's find the derivative first:

$$y' = 2(4x^3 + 3)(12x^2) = (8x^3 + 6)(12x^2)$$

$$y' = 96x^5 + 72x^2$$

$$m = y'(-1) = 96(-1)^5 + 72(-1)^2 = -24$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

1. $y = (4x^3 + 3)^2$ at $(-1,1)$

With a slope of $m = -24$ we now write the equation of the tangent line:

$$y - 1 = -24\big(x - (-1)\big)$$
$$y - 1 = -24x - 24$$
$$y = -24x - 23$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

2. $f(x) = tan^2(x)$ at $\left(\frac{\pi}{4}, 1\right)$

Recall,

$$f(x) = tan^2(x) = (\tan x)^2$$

So

$$f'(x) = 2(\tan x) \cdot sec^2(x)$$

Now,

$$m = f'\left(\frac{\pi}{4}\right) = 2\tan\left(\frac{\pi}{4}\right) sec^2\left(\frac{\pi}{4}\right)$$
$$= 2 \cdot 1 \cdot 2 = 4$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

2. $f(x) = tan^2(x)$ at $\left(\frac{\pi}{4}, 1\right)$

With $m = 4$, we have

$$y - 1 = 4\left(x - \frac{\pi}{4}\right)$$
$$y - 1 = 4x - \pi$$
$$y = 4x - \pi + 1$$

The End

- Thank you to Desmos.com and their free online graphing utility for the graphs.