# Implicit Differentiation

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# Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

# Calculus
## Differentiation

Implicit Differentiation

# Terminology

- An explicit function is a function that explicitly tells you how to find one of the variable values such as $y = f(x)$.
- An implicit function is less direct in that no variable has been isolated an in many cases it cannot be isolated. An example might be $xy = 6$ or $x^2 - xy + y^2 - 4 = 0$. In the first example, we could isolate either variable easily. In the second example it is not easy to isolate either variable (possible, but not easy).

# Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation with respect to *x*. This will use the chain rule for all terms involving *y*.
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$.

# Examples: Find $\frac{dy}{dx}$ by implicit differentiation.

*1.* $x^2 - y^2 = 25$

Always remember that $y = y(x)$. So anytime you take the derivative of $y$ you will get $y' = \frac{dy}{dx}$.

- The derivative of $x^2$ is $2x$
- The derivative of $y^2$ is $2y\frac{dy}{dx}$ (chain rule)
- The derivative of 25 is 0

Examples: Find $\frac{dy}{dx}$ by implicit differentiation.

*1.* $x^2 - y^2 = 25$

Derivative is

$$2x - 2y\frac{dy}{dx} = 0$$

Now solve,

$$-2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$$

Examples: Find $\frac{dy}{dx}$ by implicit differentiation.

*2.* $x^3 + y^3 = 64$

Derivative is

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

Solving,

$$3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

# Examples: Find $\frac{dy}{dx}$ by implicit differentiation.

*3.* $x^2y + y^2x = -2$

We will have two uses of the product rule:

Derivative of $x^2y$ is $x^2\frac{dy}{dx} + y(2x)$

Derivative of $y^2x$ is $y^2(1) + x(2y\frac{dy}{dx})$

Together this gives:

$$x^2\frac{dy}{dx} + y(2x) + y^2(1) + x\left(2y\frac{dy}{dx}\right) = 0$$

Examples: Find $\frac{dy}{dx}$ by implicit differentiation.

3. $x^2y + y^2x = -2$

$$x^2\frac{dy}{dx} + y(2x) + y^2(1) + x\left(2y\frac{dy}{dx}\right) = 0$$

$$x^2\frac{dy}{dx} + 2xy + y^2 + 2xy\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$(x^2 + 2xy)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

Examples: Find $\frac{dy}{dx}$ by implicit differentiation.

*4.* $\cot y = x - y$

$$-csc^2 y\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - csc^2 y\frac{dy}{dx} = 1$$

$$(1 - csc^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{1 - csc^2 y}$$

Examples: Find TWO explicit functions by solving the equation for *y* in terms of *x*.

*1.* $x^2 + y^2 = 64$

Explicit functions are of the form y =

$$y^2 = 64 - x^2$$

$$y = +\sqrt{64 - x^2} \quad and \quad y = -\sqrt{64 - x^2}$$

To find $\frac{dy}{dx}$ at this point, we would have to find two derivatives. Implicit makes it easier.

Examples: Find TWO explicit functions by solving the equation for *y* in terms of *x*.

*2.* $16y^2 - x^2 = 16$

$$16y^2 = 16 + x^2$$

$$y^2 = \frac{16 + x^2}{16}$$

$$y = \pm\sqrt{\frac{16 + x^2}{16}}$$

$$y = \frac{\sqrt{16 + x^2}}{4} \quad and \quad y = \frac{-\sqrt{16 + x^2}}{4}$$

Examples: Find *dy/dx* by implicit differentiation and evaluate the derivative at the given point.

*1.* $xy = 6, \quad (-6, -1)$

Product rule: $x\frac{dy}{dx} + y(1) = 0$

Solve: $x\frac{dy}{dx} = -y$ becomes $\frac{dy}{dx} = \frac{-y}{x}$

Evaluate: $\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-(-1)}{-6} = -\frac{1}{6}$

Examples: Find *dy/dx* by implicit differentiation and evaluate the derivative at the given point.

*2.* $(x+y)^3 = x^3 + y^3 \quad (-1,1)$

Derivative:

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x+y)^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 3x^2 - 3(x+y)^2$$

$$\frac{dy}{dx} = \frac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2}$$

Examples: Find *dy/dx* by implicit differentiation and evaluate the derivative at the given point.

*2.* $(x+y)^3 = x^3 + y^3 \quad (-1,1)$

$$\frac{dy}{dx} = \frac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{3(-1)^2 - 3(-1+1)^2}{3(-1+1)^2 - 3(1)^2} = \frac{3-0}{0-3} = -1$$

Examples: Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

Fact: To find the second derivative, we must first find the first derivative!

*1.* $x^2y^2 - 2x = 3$

Derivative using product rule for first term:

$$x^2\left(2y\frac{dy}{dx}\right) + y^2(2x) - 2 = 0$$

$$2x^2y\frac{dy}{dx} = 2 - 2xy^2$$

$$\frac{dy}{dx} = \frac{2 - 2xy^2}{2x^2y} = \frac{1 - xy^2}{x^2y}$$

# Examples: Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

To find $\frac{d^2y}{dx^2}$ we use the quotient rule with some product rule thrown in on $\frac{dy}{dx} = \frac{1-xy^2}{x^2y}$:

Der of numerator: $0 - \left(x2y\frac{dy}{dx} + y^2(1)\right)$

Der of denominator: $x^2\frac{dy}{dx} + y(2x)$

$$\frac{d^2y}{dx^2} = \frac{x^2y\left(-2xy\frac{dy}{dx} - y^2\right) - (1 - xy^2)(x^2\frac{dy}{dx} + 2xy)}{(x^2y)^2}$$

## Examples: Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

$$\frac{d^2y}{dx^2} = \frac{-x^2y\left(2xy\frac{dy}{dx} + y^2\right) - (1 - xy^2)(x^2\frac{dy}{dx} + 2xy)}{(x^2y)^2}$$

Cleaning it up a bit:

$$\frac{d^2y}{dx^2}$$

$$= \frac{-x^2y\left(2xy\frac{1 - xy^2}{x^2y} + y^2\right) - (1 - xy^2)(x^2\frac{1 - xy^2}{x^2y} + 2xy)}{x^4y^2}$$

Examples: Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

*2.* $1 - xy = x - y$

Derivative:

$$0 - \left( x\frac{dy}{dx} + y(1) \right) = 1 - \frac{dy}{dx}$$

$$-x\frac{dy}{dx} + \frac{dy}{dx} = 1 + y$$

$$\frac{dy}{dx} = \frac{1+y}{1-x}$$

Examples: Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

*2.* $1 - xy = x - y$

$$\frac{dy}{dx} = \frac{1+y}{1-x}$$

Second derivative:

$$y'' = \frac{(1-x)\frac{dy}{dx} - (1+y)(-1)}{(1-x)^2}$$

$$y'' = \frac{(1-x)\frac{1+y}{1-x} - (1+y)(-1)}{(1-x)^2} = \frac{2+2y}{(1-x)^2}$$

Examples: Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

*3.* $y^2 = 10x$

Derivative: $2y\frac{dy}{dx} = 10$ so $\frac{dy}{dx} = \frac{10}{2y} = \frac{5}{y}$

Second derivative:

$$y'' = \frac{y(0) - 5\frac{dy}{dx}}{y^2} = \frac{-5\frac{5}{y}}{y^2} = \frac{-25}{y^3}$$

# The End