Related Rates

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Suggested Review Topics

- Algebra skills reviews suggested:
  - Pythagorean Theorem
  - Similar Triangles (Geometry)
  - Area and volume formulas
  - Distance formula
- Trigonometric skills reviews suggested:
  - None

Math 1411
Differentiation

Related Rates

Special Note

- This section also has a video available on my YouTube channel. You can find it at https://www.youtube.com/watch?v=MIWA6C2y9AU
- This particular video takes you through several homework problems from WebAssign as well as gives some tips and tricks not presented here.

Guidelines for Solving Related Rates Problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities, if needed.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Use the Chain Rule to implicitly differentiate both sides of the equation with respect to time t.
4. After completing step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $\frac{dx}{dt} = 2$ centimeters per second.

- Find rate of change of the distance tells us to find $\frac{dd}{dt}$.
- Origin is (0,0) and a point on the graph is $(x, y) \ = (x, x^2 + 1)$
- Distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $\frac{dx}{dt} = 2$ centimeters per second.

- Substituting:

$$d = \sqrt{(x-0)^2 + (x^2+1-0)^2}$$
$$= \sqrt{x^2 + (x^2+1)^2}$$

- Implicit differentiation:

$$\frac{dd}{dt} = \frac{1}{2}(x^2 + (x^2+1)^2)^{-\frac{1}{2}}(2x\frac{dx}{dt} + 2(x^2+1)2x\frac{dx}{dt})$$

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- Implicit differentiation:

$$\frac{dd}{dt} = \frac{1}{2}\left(x^2 + (x^2+1)^2\right)^{-\frac{1}{2}}\left(2x\frac{dx}{dt} + 2(x^2+1)2x\frac{dx}{dt}\right)$$

- Simplify and use $\frac{dx}{dt} = 2$:

$$\frac{dd}{dt} = \frac{1}{2\sqrt{x^2 + (x^2+1)^2}}\left(2x(2) + 2(x^2+1)2x(2)\right)$$

$$\frac{dd}{dt} = \frac{1}{2\sqrt{x^2 + (x^2+1)^2}}\left(4x + 8x(x^2+1)\right) = \frac{4x^3 + 6x}{\sqrt{x^2 + (x^2+1)^2}}$$

Example: The radius *r* of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ cm and (b) $r = 32$ cm.

- Area of a circle is $A = \pi r^2$
- Rate of change of area is $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$
- Radius increasing is $\frac{dr}{dt} = 4$ cm/min

$$a)\ \frac{dA}{dt} = 2\pi(8)(4) = 64\pi\ cm^2/min$$

Example: The radius *r* of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ cm and (b) $r = 32$ cm.

- Area of a circle is $A = \pi r^2$
- Rate of change of area is $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$
- Radius increasing is $\frac{dr}{dt} = 4$ cm/min

$$b)\ \frac{dA}{dt} = 2\pi(32)(4) = 256\pi\ cm^2/min$$

Example. Let *A* be the area of a circle of radius *r* that is changing with respect to time. If *dr/dt* is constant, is *dA/dt* constant? Explain.

From the previous example we saw that $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. When the rate of change of the radius is constant, we can still get other values for rate of change of the area because $\frac{dA}{dt}$ depends on both $\frac{dr}{dt}$ and the length *r* of the radius itself. Notice that even though $\frac{dr}{dt} = 4$ cm/min in the previous example, we found different answers for $\frac{dA}{dt}$.

Example: The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rates of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when $r = 6$ inches and $r = 24$ inches.

- When doing related rates problems, try to eliminate variables if possible.
  - With $h = 3r, \quad V = \frac{1}{3}\pi r^2(3r) = \pi r^3$
  - If $V = \pi r^3$, then $\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

$$a)\ \frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi\ in^3/min$$

Example: The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rates of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when $r = 6$ inches and $r = 24$ inches.

- When doing related rates problems, try to eliminate variables if possible.
  - With $h = 3r$, $\quad V = \frac{1}{3}\pi r^2(3r) = \pi r^3$
  - If $V = \pi r^3$, then $\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

b) $\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi\ in^3/min$

A construction worker pulls a five-meter plank up the side of a building under construction by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meters per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

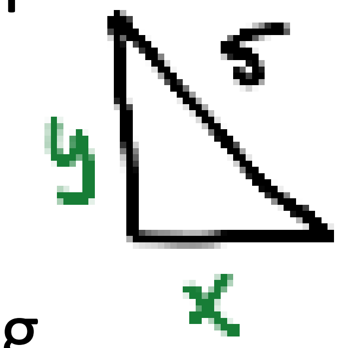
With $\frac{dy}{dt} = 0.15$

Want to find $\frac{dx}{dt}$

How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

- Our first step is to write the relationship equation. We will use this twice.

$$x^2 + y^2 = 5^2$$



- The first use will be to solve for $\frac{dx}{dt}$ using implicit differentiation.
- The second use will be to find a missing value not explicitly given in the statement of the problem.

How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

$$x^2 + y^2 = 5^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$$

We know dy/dt and x, but what is y?

How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

$$x^2 + y^2 = 5^2$$

With $x = 2.5$, gives

$$(2.5)^2 + y^2 = 25$$
$$6.25 + y^2 = 25$$
$$y^2 = 18.75$$
$$y = \sqrt{18.75}$$

Now we can answer the question.

The worker pulls the rope at a rate of 0.15 meters per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

$$\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{-\sqrt{18.75}}{2.5}(0.15)$$

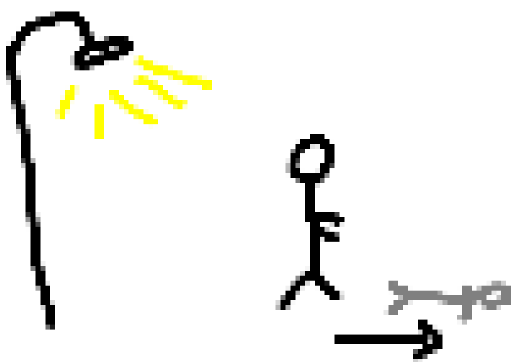
$$\frac{dx}{dt} \approx -0.26\, m/s$$

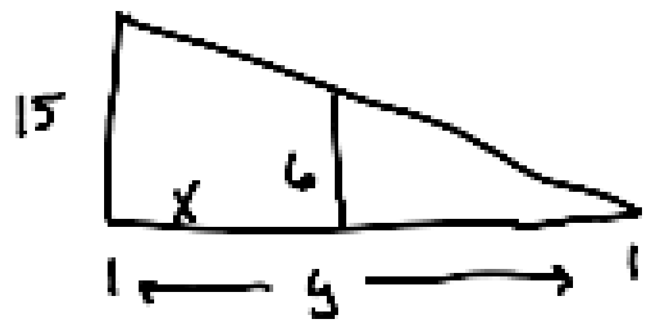
The negative sign indicates direction.

A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground. When he is 10 feet from the base of the light,

a) at what rate is the tip of his shadow moving?

b) at what rate is the length of his shadow changing?





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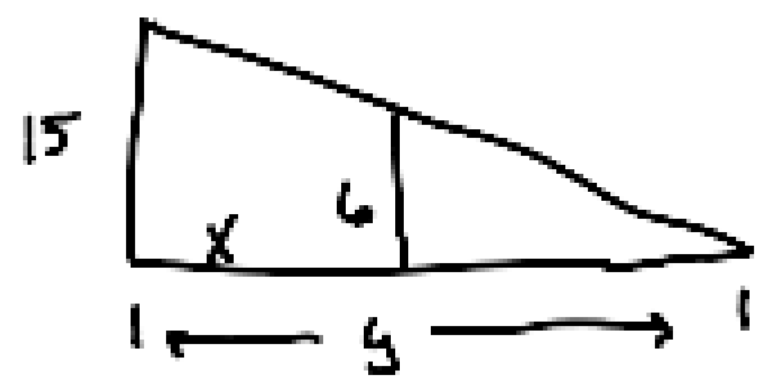
- $\frac{15}{6} = \frac{y}{y-x}$
- $15y - 15x = 6y$
- $9y = 15x$
- $y = \frac{15x}{9} = \frac{5x}{3}$



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a) at what rate is the tip of his shadow moving?

a) $y = \frac{5x}{3}$ with $\frac{dx}{dt} = 5$ ft/sec and $x = 10$ feet:

$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \approx 8.33\,\frac{\text{ft}}{\text{sec}}$$

b) The length of the shadow is $y - x$ so

$$\frac{d}{dt}(y - x) = \frac{dy}{dt} - \frac{dx}{dt} = 8.33 \; - 5 = 3.33\,\frac{\text{ft}}{\text{sec}}$$

The End