Extrema on an Interval

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Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Math 1411
Applications of Differentiation

Extrema on an Interval

Note

- There is a video that accompanies this section on my website and on YouTube. The link is: https://www.youtube.com/watch?v=7DfSX97tfJg
- In this video I talk specifically about Critical Numbers and give several examples.

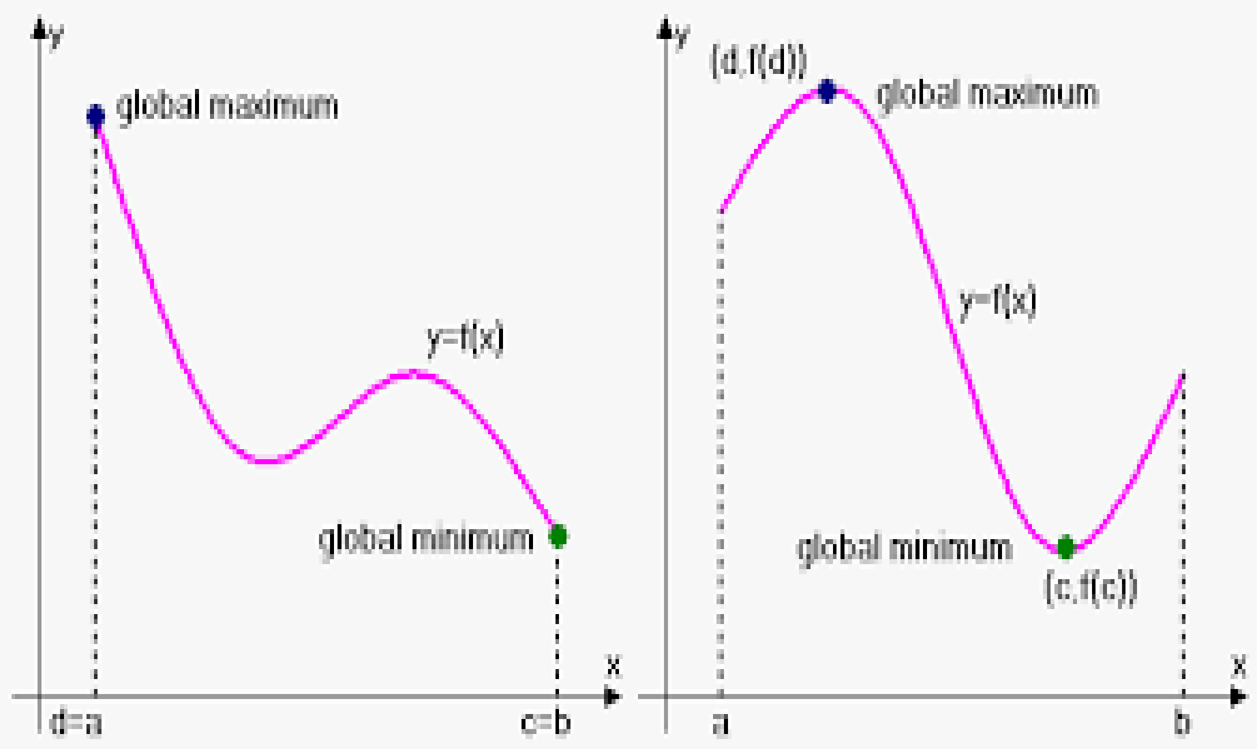
Definition of Extrema

- Let *f* be defined on an interval *I* containing *c*.
  1. $f(c)$ is the minimum of *f* on *I* if $f(c) \leq f(x)$ for all *x* in *I.*
  2. $f(c)$ is the maximum of *f* on *I* if $f(c) \geq f(x)$ for all *x* in *I.*

- The minimum and maximum of a function on an interval are the extreme values, or extrema, of the function on the interval.
- The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or global minimum and global maximum, on the interval.

The Extreme Value Theorem

- If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.



Definition of Relative Extrema

1. If there is an open interval containing *c* on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of *f*, or you can say that *f* has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing *c* on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum of *f*, or you can say that *f* has a relative minimum at $(c, f(c))$.

- The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima.
- Relative maximum and relative minimum are sometimes called local maximum and local minimum, respectively.

Critical Numbers

- Definition – Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f

- Theorem – If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f. That is, relative extrema occur only at critical numbers.

Guidelines for Finding Extrema on a Closed Interval

- To find extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.
  1. Find the critical numbers of f in (a, b). Note: These are *x* values where the derivative is 0 or does not exist.
  2. Evaluate f at each critical number in (a, b).
  3. Evaluate f at each endpoint of $[a, b]$.
  4. The least of these values is the minimum. The greatest is the maximum.

Examples: Find the derivative of the function at the indicated extremum.

1. $f(x) = \cos(\frac{\pi x}{2})$ at $(0,1)$ and at $(2,-1)$

The derivative is $f'(x) = -\frac{\pi}{2}\sin(\frac{\pi x}{2})$

At $(0,1)$: $f'(0) = -\frac{\pi}{2}\sin(0) = 0$

At $(2,-1)$: $f'(2) = -\frac{\pi}{2}\sin(\pi) = 0$

Derivative is zero at the extrema, no surprise here.

Examples: Find the derivative of the function at the indicated extremum.

2. $f(x) = -3x\sqrt{x+1}$ at $(-1,0)$ and at $(-\frac{2}{3}, \frac{2\sqrt{3}}{3})$

The derivative using the product and chain rules:

$$f'(x) = (-3x)\frac{1}{2}(x+1)^{-\frac{1}{2}}(1) + \left(\sqrt{x+1}\right)(-3)$$

$$f'(x) = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$$

Examples: Find the derivative of the function at the indicated extremum.

2. $f(x) = -3x\sqrt{x+1}$ at $(-1,0)$

$$f'(x) = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$$

At $(-1,0)$: $f'(-1) = \frac{-3(-1)}{2\sqrt{-1+1}} - 3\sqrt{-1+1}$

The fraction has a zero denominator and a non-zero numerator and so the derivative is undefined at $x = -1$.

Examples: Find the derivative of the function at the indicated extremum.

2. $f(x) = -3x\sqrt{x+1}$ at $(-\frac{2}{3}, \frac{2\sqrt{3}}{3})$

$$f'(x) = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$$

At $(-\frac{2}{3}, \frac{2\sqrt{3}}{3})$: $f'\left(\frac{2}{3}\right) = \frac{-3(\frac{-2}{3})}{2\sqrt{\frac{-2}{3}+1}} - 3\sqrt{\frac{-2}{3}+1}$

$$= \frac{2}{2\sqrt{\frac{1}{3}}} - 3\sqrt{\frac{1}{3}} = \sqrt{3} - \sqrt{3} = 0$$

Examples: Locate the absolute extrema of the function on the closed interval.

1. $f(x) = \frac{2x+5}{3}$ on $[0,5]$

Rewrite: $f(x) = \frac{2}{3}x + \frac{5}{3}$

Derivative: $f'(x) = \frac{2}{3}$

Solve: The derivative is always defined, never zero.

Conclude: Only possible extrema is at the endpoints.

Evaluate: $f(0) = \frac{5}{3}$ and $f(5) = 5$.

Solution: Minimum at $(0, \frac{5}{3})$ and maximum at $(5,5)$

Examples: Locate the absolute extrema of the function on the closed interval.

2. $f(x) = x^3 - 12x$ on [0,4]

Derivative: $f'(x) = 3x^2 - 12$

Solve: $0 = 3x^2 - 12$ adding 12 and dividing by 3 gives $x^2 = 4$ and so $x = \pm 2$.

Test endpoints and critical numbers:

$f(0) = 0$	not max or min
$f(2) = -16$	minimum
$f(4) = 16$	maximum
$f(-2)$ not in interval	

Examples: Locate the absolute extrema of the function on the closed interval.

3. $g(x) = \sqrt[3]{x}$ on [-1,1]

Rewrite: $g(x) = x^{1/3}$

Derivative: $g'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

Solve: The derivative is never zero. Why?

Conclude: The derivative is undefined at x = 0.

Test endpoints and critical numbers:

$g(-1) = -1$ minimum

$g(0) = 0$

$g(1) = 1$ maximum

Examples: Locate the absolute extrema of the function on the closed interval.

4. $h(t) = \frac{t}{t-2}$ on [3,5]

Derivative:

$$h'(t) = \frac{(t-2)(1) - t(1)}{(t-2)^2} = \frac{t-2-t}{(t-2)^2} = \frac{-2}{(t-2)^2}$$

The derivative is undefined at $t = 2$, not in interval, and the derivative is never zero → no critical numbers.

$h(3) = 3$ maximum

$h(5) = \frac{5}{3}$ minimum

Examples: Locate the absolute extrema of the function on the closed interval.

5. $g(x) = \sec x \quad [-\frac{\pi}{6}, \frac{\pi}{3}]$

Derivative: $g'(x) = \sec x \tan x = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \frac{\sin x}{cos^2 x}$

The derivative is undefined when $\cos x = 0$, so at $x = \frac{\pi}{2} + k\pi$. But these values are not in our interval.

The derivative is zero when $\sin x = 0$ so at x = 0 for this interval. This is a critical number.

Examples: Locate the absolute extrema of the function on the closed interval.

5. $g(x) = \sec x \quad [-\frac{\pi}{6}, \frac{\pi}{3}]$

Derivative: $g'(x) = \sec x \tan x = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \frac{\sin x}{cos^2 x}$

$$g\left(-\frac{\pi}{6}\right) = \sec(-\frac{\pi}{6}) = \frac{2\sqrt{3}}{3} \approx 1.1547$$

$$g(0) = \sec(0) = 1 \qquad \text{MIN}$$

$$g\left(\frac{\pi}{3}\right) = \sec(\frac{\pi}{3}) = 2 \qquad \text{MAX}$$

End of Lecture