Rolle’s Theorem and the Mean Value Theorem

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Suggested Review Topics

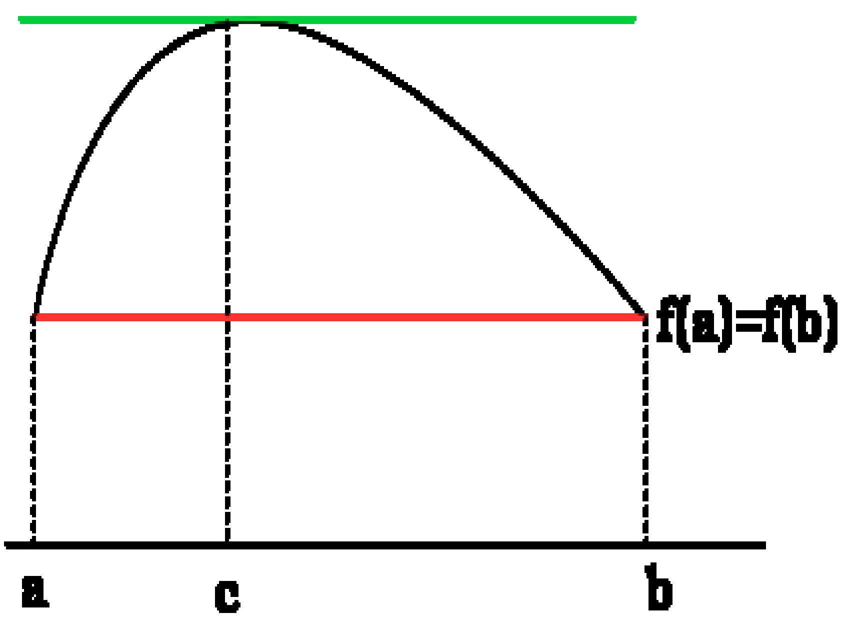
- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem

- Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there is at least one number c in (a, b) such that $f'(c) = 0$.



Examples: Find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the x-intercepts.

1. $f(x) = x(x - 3)$

The x-intercepts occur when $0 = x(x - 3)$ so at $x = 0$ and $x = 3$.

Derivative: $f'(x) = x(1) + (x - 3)(1) = 2x - 3$

Solve: $0 = 2x - 3$ becomes $3 = 2x$ and so $x = 2/3$.

The derivative is zero at 2/3 and 2/3 is between the x-intercepts of $x = 0$ and $x = 3$.

Examples: Find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the x-intercepts.

2. $f(x) = -3x\sqrt{x+1}$

The x-intercepts occur when $0 = -3x\sqrt{x+1}$ so at $x = 0$ and $x = -1$.

Derivative:

$$f'(x) = -3x\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}\right) + \sqrt{x+1}(-3).$$

That is, $f'(x) = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$

Examples: Find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the x-intercepts.

2. $f(x) = -3x\sqrt{x+1}$

Solve: $0 = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$

$$3\sqrt{x+1} = \frac{-3x}{2\sqrt{x+1}}$$

$$(2\sqrt{x+1})3\sqrt{x+1} = \frac{-3x}{\cancel{2\sqrt{x+1}}}(\cancel{2\sqrt{x+1}})$$

$$6(x+1) = -3x$$

$$6 = -9x \;\; so \;\; x = -2/3$$

-2/3 is between -1 and 0.

Examples: Determine whether Rolle's Theorem can be applied to f on the closed interval. If RT can be applied, find all values c in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

1. $f(x) = x^2 - 5x + 4$, [1,4]

If RT can be applied we must check:

continuous on [1,4] -> yes, polynomial

differentiable on [1,4] -> yes, polynomial

$f(1) = f(4)$ -> yes, $f(1) = 0$ and $f(4) = 0$

Now we can work on finding the value c.

Examples: Determine whether Rolle's Theorem can be applied to f on the closed interval. If RT can be applied, find all values c in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

1. $f(x) = x^2 - 5x + 4$, [1,4]

Derivative: $f'(x) = 2x - 5$

Solve: $f'(c) = 0$ gives $0 = 2c - 5$ so $c = 5/2$

Examples: Determine whether Rolle's Theorem can be applied to *f* on the closed interval. If RT can be applied, find all values *c* in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

2. $f(x) = x^{2/3} - 1$, [-8,8]

continuous: yes

differentiable: ??

Derivative: $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

differentiable: not differentiable at $x = 0$

Rolle's Theorem does not apply

Examples: Determine whether Rolle's Theorem can be applied to f on the closed interval. If RT can be applied, find all values c in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

3. $f(x) = \frac{x^2-2x-3}{x+2}, \ [-1,3]$

Derivative:

$$f'(x) = \frac{(x+2)(2x-2)-(x^2-2x-3)(1)}{(x+2)^2}$$

$$= \frac{2x^2-2x+4x-4-x^2+2x+3}{(x+2)^2}$$

$$= \frac{x^2+4x-1}{(x+2)^2}$$

Examples: Determine whether Rolle's Theorem can be applied to *f* on the closed interval. If RT can be applied, find all values *c* in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

3. $f(x) = \frac{x^2-2x-3}{x+2}, \ [-1,3]$

Derivative: $f'(x) = \frac{x^2+4x-1}{(x+2)^2}$

Not continuous or differentiable at x = -2, not in our interval so no worries.

$f(-1) = 0$ and $f(3) = 0$

Rolle's Theorem applies, we can find c.

Examples: Determine whether Rolle's Theorem can be applied to *f* on the closed interval. If RT can be applied, find all values *c* in the open interval such that $f'(c) = 0$.

3. $f(x) = \frac{x^2-2x-3}{x+2}$, $[-1,3]$

Derivative: $f'(x) = \frac{x^2+4x-1}{(x+2)^2}$

Solve: $f'(c) = 0$ gives $0 = \frac{c^2+4c-1}{(c+2)^2}$.

A rational expression is zero when the numerator is zero so $0 = c^2 + 4c - 1$. Using the quadratic formula we find $c = -2 \pm \sqrt{5}$.

However, $c = -2 + \sqrt{5} \approx 0.236$ is in the interval.

Examples: Determine whether Rolle's Theorem can be applied to f on the closed interval. If RT can be applied, find all values c in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

4. $f(x) = \frac{x^2-1}{x}, \ [-1,1]$

Not continuous (or differentiable) at x = 0 so RT does not apply.

Examples: Determine whether Rolle's Theorem can be applied to f on the closed interval. If RT can be applied, find all values c in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

5. $g(x) = \cos x\,, [0,2\pi]$

We know this is continuous and differentiable everywhere.

We know that $\cos 0 = \cos 2\pi$

Rolle's Theorem applies:

Derivative: $f'(x) = -\sin x$

Examples: Determine whether Rolle's Theorem can be applied to f on the closed interval. If RT can be applied, find all values c in the open interval such that $f'(c) = 0$. If RT cannot be applied, explain why not.

5. $g(x) = \cos x, [0,2\pi]$

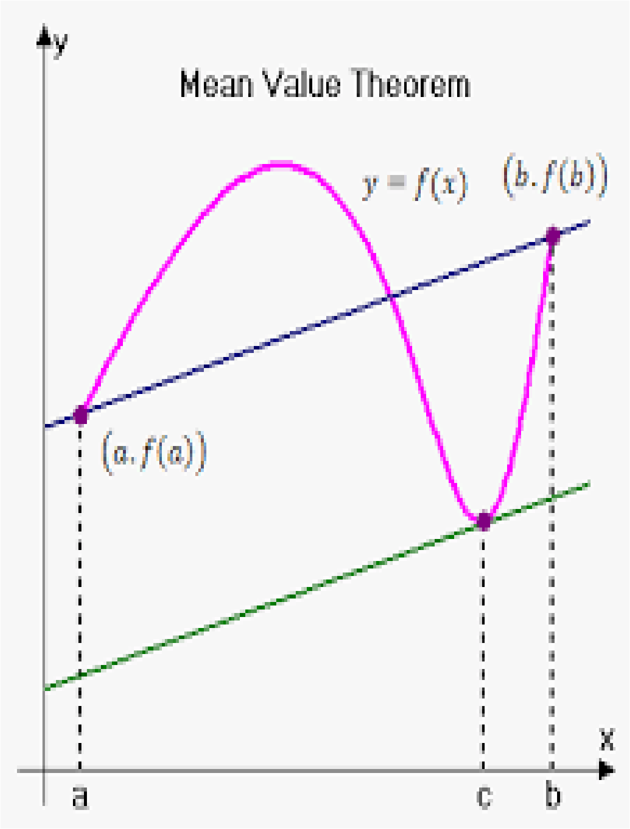
Derivative: $f'(x) = -\sin x$

Solve: $0 = -\sin c$ when $c = 0,\ \pi,\ 2\pi.$

All three are in the interval, but the theorem states that c will be in the open interval so the only value we can keep is $c = \pi$.

The Mean Value Theorem

- If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b)such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Examples: Determine whether the MVT can be applied to f on the closed interval. If the MVT can be applied, find all values of c given by the theorem. If the MVT cannot be applied, explain why not.

1. $f(x) = x^3$, [0,1]

Continuous and differentiable everywhere, MVT applies.

Evaluate endpoints: $f(0) = 0 \ and \ f(1) = 1$

Derivative: $f'(x) = 3x^2$

Set up the MVT:

$$f'(c) = 3c^2 = \frac{1-0}{1-0} = \frac{f(b) - f(a)}{b - a}$$

Examples: Determine whether the MVT can be applied to *f* on the closed interval. If the MVT can be applied, find all values of *c* given by the theorem. If the MVT cannot be applied, explain why not.

1. $f(x) = x^3$, [0,1]

$$3c^2 = \frac{1-0}{1-0}$$

$$3c^2 = 1$$

$$c^2 = \frac{1}{3} \quad so \quad c = \pm\sqrt{\frac{1}{3}}$$

Our answer is $c = \frac{\sqrt{3}}{3}$ after simplifying and considering the interval given.

Examples: Determine whether the MVT can be applied to f on the closed interval. If the MVT can be applied, find all values of c given by the theorem. If the MVT cannot be applied, explain why not.

2. $f(x) = x^4 - 8x,\ [0,2]$

Continuous and differentiable everywhere.

Evaluate: $f(0) = 0\ \ and\ f(2) = 0.$

Note: Rolle's Theorem also applies.

Derivative: $f'(x) = 4x^3 - 8$

Solve:

$$4c^3 - 8 = \frac{0-0}{2-0} = 0$$

Examples: Determine whether the MVT can be applied to *f* on the closed interval. If the MVT can be applied, find all values of *c* given by the theorem. If the MVT cannot be applied, explain why not.

2. $f(x) = x^4 - 8x,\ [0,2]$

Solve:

$$4c^3 - 8 = \frac{0-0}{2-0} = 0$$

$$4c^3 = 8\ becomes\ c^3 = 2$$

$$c = \sqrt[3]{2}$$

Examples: Determine whether the MVT can be applied to f on the closed interval. If the MVT can be applied, find all values of c given by the theorem. If the MVT cannot be applied, explain why not.

3. $f(x) = \frac{x+1}{x},\ [-1,2]$

This function is not continuous, nor differentiable, at $x = 0$ so MVT does NOT apply.

Examples: Determine whether the MVT can be applied to f on the closed interval. If the MVT can be applied, find all values of c given by the theorem. If the MVT cannot be applied, explain why not.

4. $f(x) = \sin x\,,\ [0, \pi]$

Continuous and differentiable everywhere.

Evaluate: $f(0) = 0\ \ and\ \ f(\pi) = 0$

Derivative: $f'(x) = \cos x$

Solve:

$$\cos c = \frac{0 - 0}{\pi - 0} = 0$$

Examples: Determine whether the MVT can be applied to f on the closed interval. If the MVT can be applied, find all values of c given by the theorem. If the MVT cannot be applied, explain why not.

4. $f(x) = \sin x$, $[0, \pi]$

Solve:

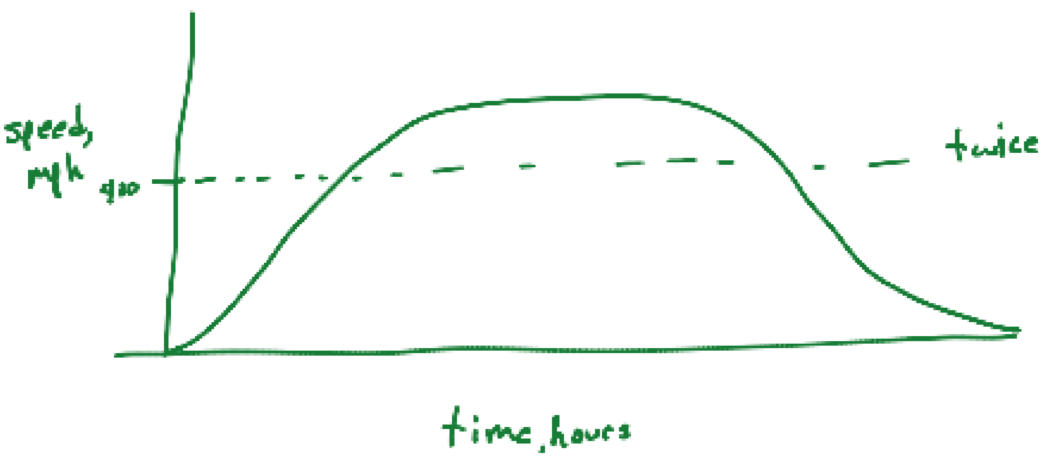
$$\cos c = \frac{0 - 0}{\pi - 0} = 0$$

Cosine is zero when $c = \frac{\pi}{2}$ for this interval.

❖ Rolle's Theorem is a special case of the Mean Value Theorem in which the endpoints are equal.

A plane begins its takeoff at 2:00 PM on a 2500 mile flight. After 5.5 hours, the plan arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 mph.

On average, the plane flew at $\frac{2500\ m}{5.5\ hr} = 454.5\ mph$. In order to *average* this speed it had to go from 0 mph up to full speed, past 454.5 mph, and then it had to power back down to land. This is the MVT.



End of Lecture