Increasing and Decreasing Functions and the First Derivative Test

By Tuesday J. Johnson

Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

Increasing and Decreasing Functions and the First Derivative Test

Note

- There is a video that accompanies this section on my website and on YouTube. The link is: https://www.youtube.com/watch?v=nUNAGTRiy64

- In this video I talk specifically about Increasing and Decreasing Functions and give several examples.

Definitions of Increasing and Decreasing Functions

- A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

- A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Theorem 3.5 (Catchy title isn't it?)

- Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Guidelines for Finding Intervals of Increase and Decrease

- Let f be continuous on the interval (a, b). To find the OPEN intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical number of f in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

Identify the open intervals on which the function is increasing or decreasing.

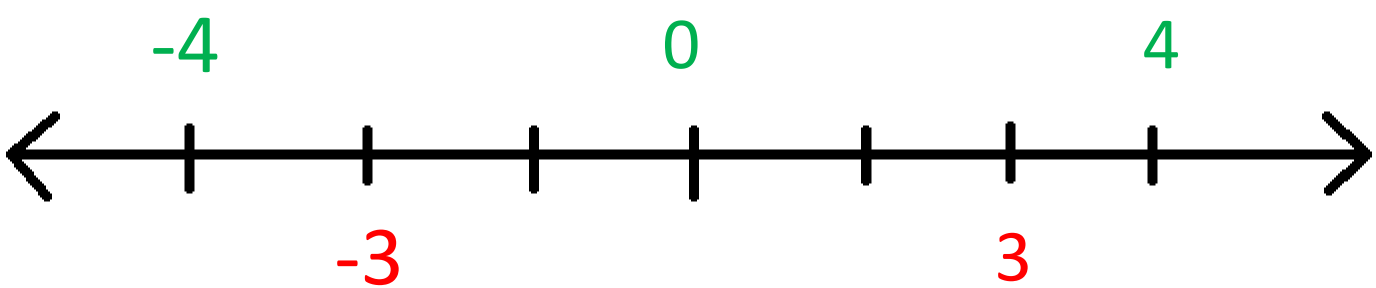
1. $h(x) = 27x - x^3$

Derivative: $h'(x) = 27 - 3x^2$

Solve: $0 = 27 - 3x^2$ becomes $3x^2 = 27$ so $x^2 = 9$ and $x = \pm 3$.

Critical numbers: $x = -3, 3$

Test Points: $x = -4,0,4$



Identify the open intervals on which the function is increasing or decreasing.

1. $h(x) = 27x - x^3$

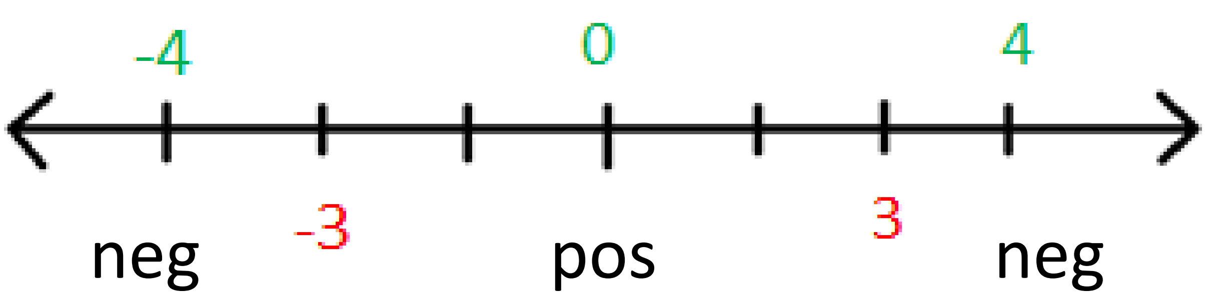
Derivative: $h'(x) = 27 - 3x^2$

Evaluate derivative at test points from each interval:

$h'(-4) = 27 - 3(-4)^2 = 27 - 3(16) =$ Negative

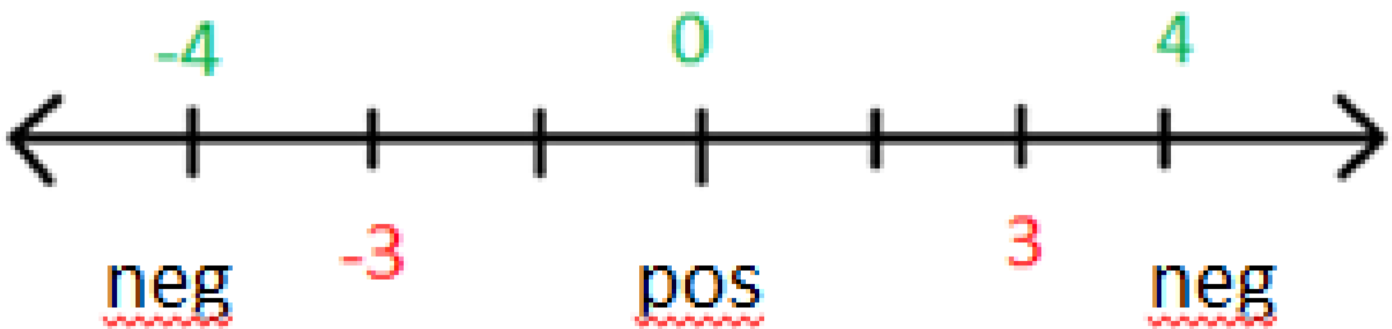
$h'(0) = 27 - 3(0)^2 = 27 =$ Positive

$h'(4) = 27 - 3(4)^2 = 27 - 3(16) =$ Negative



Identify the open intervals on which the function is increasing or decreasing.

1. $h(x) = 27x - x^3$



Conclusion:

Increasing on $(-3,3)$

Decreasing on $(-\infty, -3) \cup (3, \infty)$

Identify the open intervals on which the function is increasing or decreasing.

2. $y = x + \frac{4}{x}$

Derivative: $y' = 1 - \frac{4}{x^2}$

Critical number: $x = 0$ as derivative is undefined.

Solve: $0 = 1 - \frac{4}{x^2}$ becomes $\frac{4}{x^2} = 1$ so $x^2 = 4$ and $x = \pm 2$.

Critical numbers are $x = -2, 0, 2$ making four intervals on our number line.

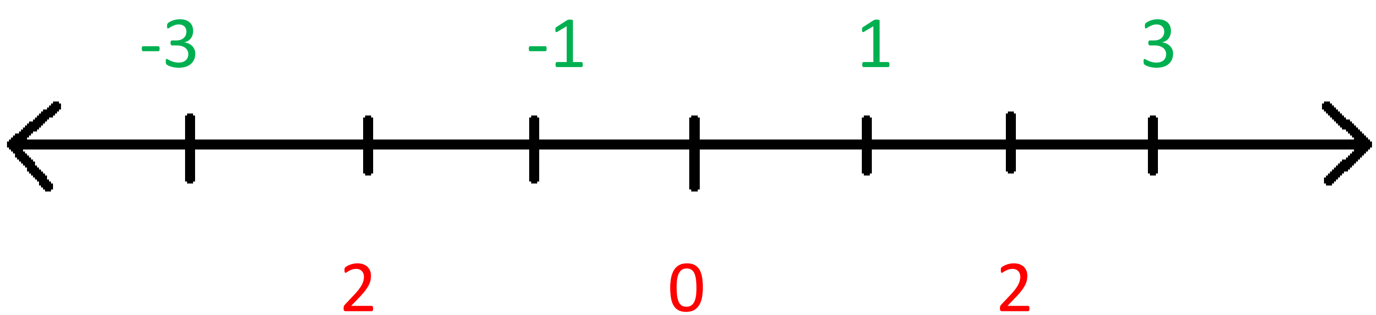
Identify the open intervals on which the function is increasing or decreasing.

2. $y = x + \frac{4}{x}$

Derivative: $y' = 1 - \frac{4}{x^2}$

Critical numbers are $x = -2, 0, 2$

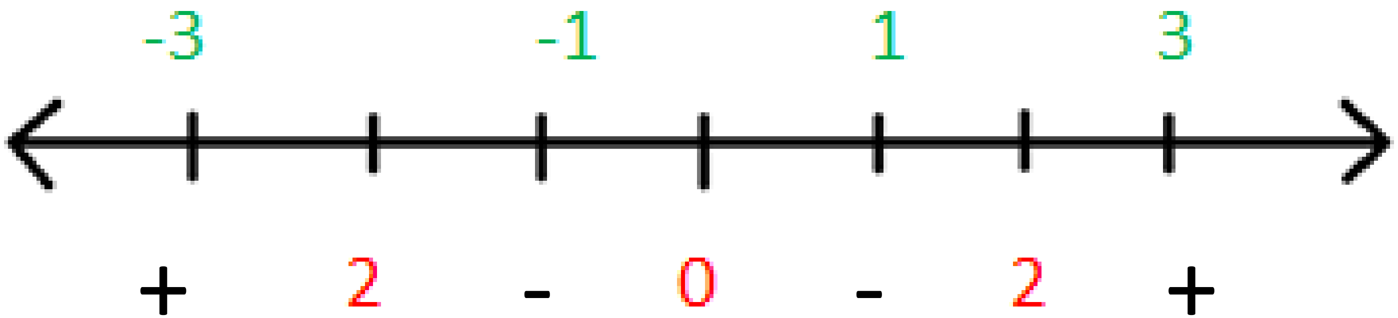
Test Points might be $x = -3, -1, 1, 3$



Identify the open intervals on which the function is increasing or decreasing.

2. $y = x + \frac{4}{x}$

Derivative: $y' = 1 - \frac{4}{x^2}$



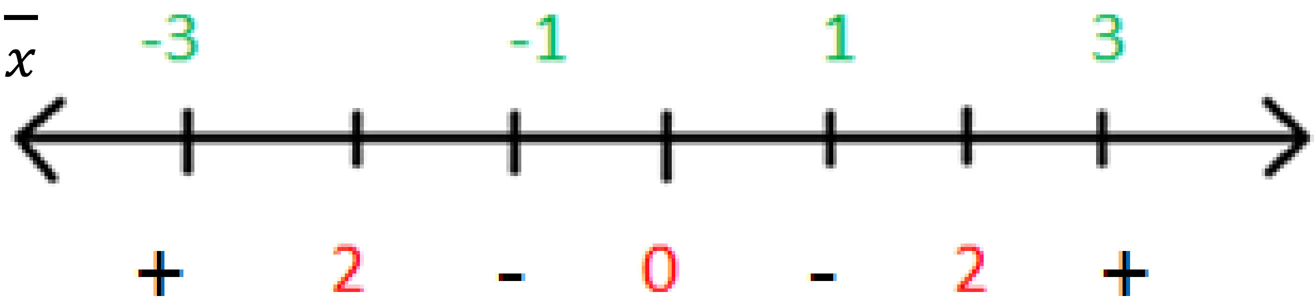
Evaluate derivative:

$$y'(-3) = 1 - \frac{4}{9} = + \qquad y'(-1) = 1 - \frac{4}{1} = -$$

$$y'(1) = 1 - \frac{4}{1} = - \qquad y'(3) = 1 - \frac{4}{9} = +$$

Identify the open intervals on which the function is increasing or decreasing.

2. $y = x + \frac{4}{x}$



Conclusion:

Increasing on $(-\infty, -2) \cup (2, \infty)$

Decreasing on $(-2,0) \cup (0,2)$

❖ We must exclude $x = 0$ in this way as the function is not defined there.

The First Derivative Test

- Let *c* be a critical number of a function *f* that is continuous on an open interval *I* containing *c*. If *f* is differentiable on the interval, except possibly at *c*, then *f(c)* can be classified as follows.

  1. If $f'(x)$ changes from negative to positive at *c*, then *f* has a relative minimum at $(c, f(c))$.
  2. If $f'(x)$ changes from positive to negative at *c*, then *f* has a relative maximum at $(c, f(c))$.
  3. If $f'(c)$ is positive on both sides of *c* or negative on both sides of *c*, then $f(c)$ is neither a relative minimum nor a relative maximum.

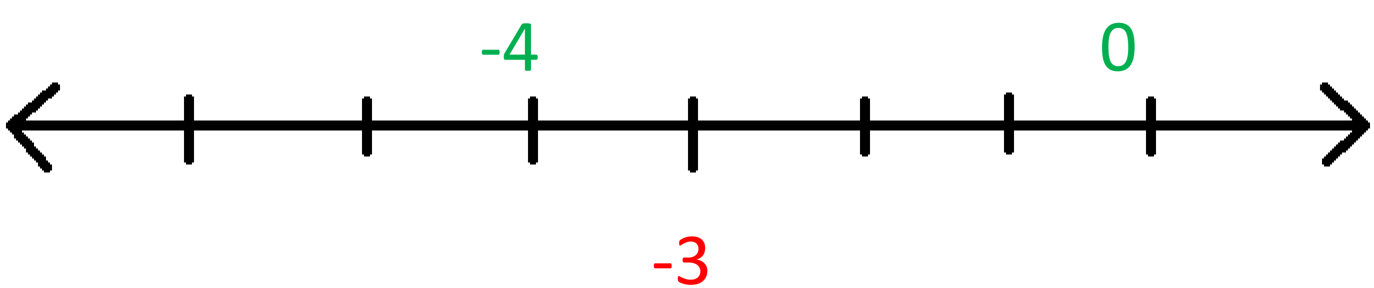
Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

1. $f(x) = x^2 + 6x + 10$

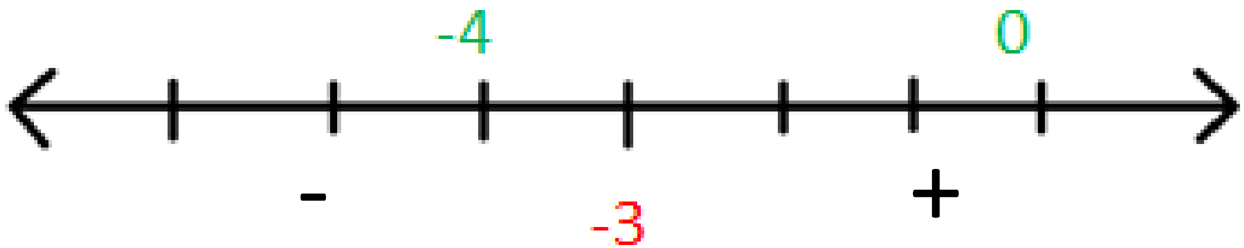
Critical numbers: $f'(x) = 0$ or $f'(x)$ DNE

$$f'(x) = 2x + 6$$
$$0 = 2x + 6$$
$$-3 = x$$

Test points might be, say, $x = -4$ and $x = 0$



Use test points in the derivative $f'(x) = 2x + 6$



$$f'(-4) = 2(-4) + 6 = negative$$

$$f'(0) = 2(0) + 6 = positive$$

Conclusion:

Increasing: $(-3, \infty)$

Decreasing: $(-\infty, -3)$

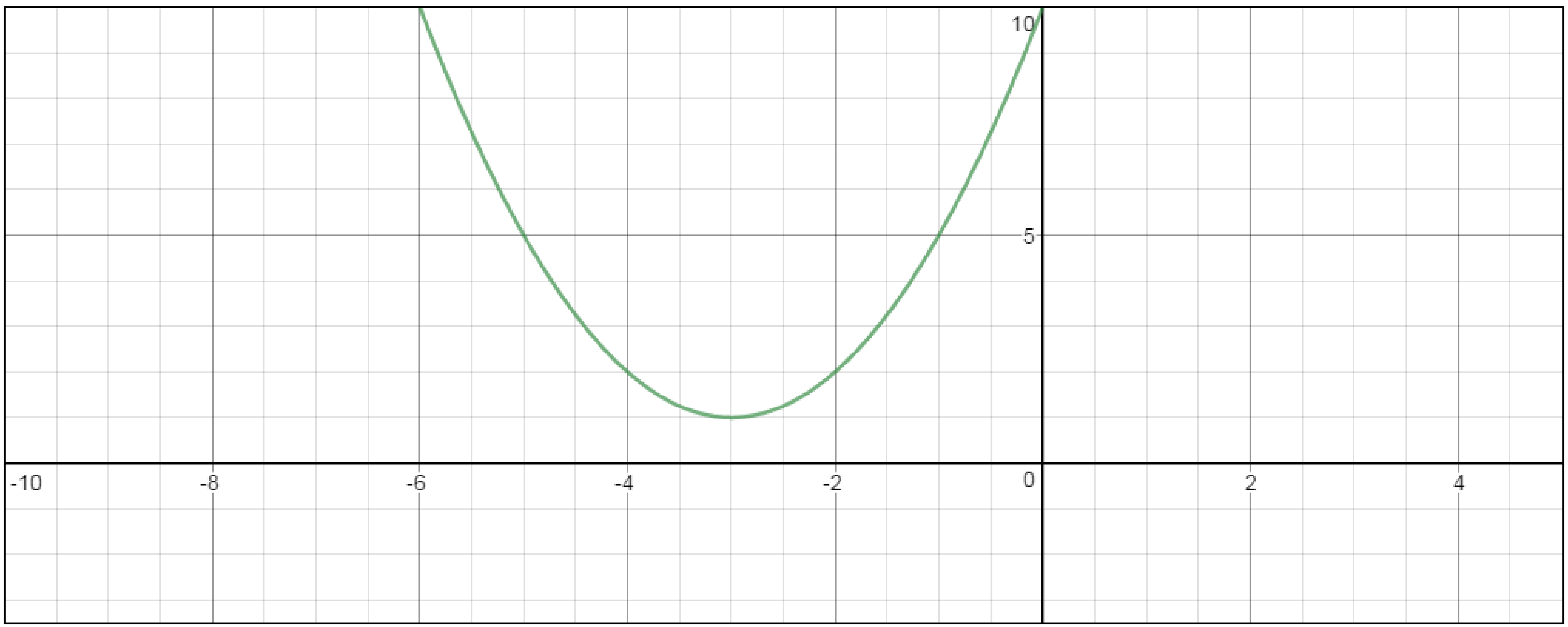
Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

1. $f(x) = x^2 + 6x + 10$

a) Critical numbers: $x = -3$
b) Increasing: $(-3, \infty)$ Decreasing: $(\infty, -3)$
c) Changes decreasing to increasing at $x = -3$ so this is the location of a minimum. The y value comes from the original function, so the minimum is the point $\left(-3, f(-3)\right) = (-3,1)$

Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

1. $f(x) = x^2 + 6x + 10$



Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

2. $f(x) = x^3 - 6x^2 + 15$

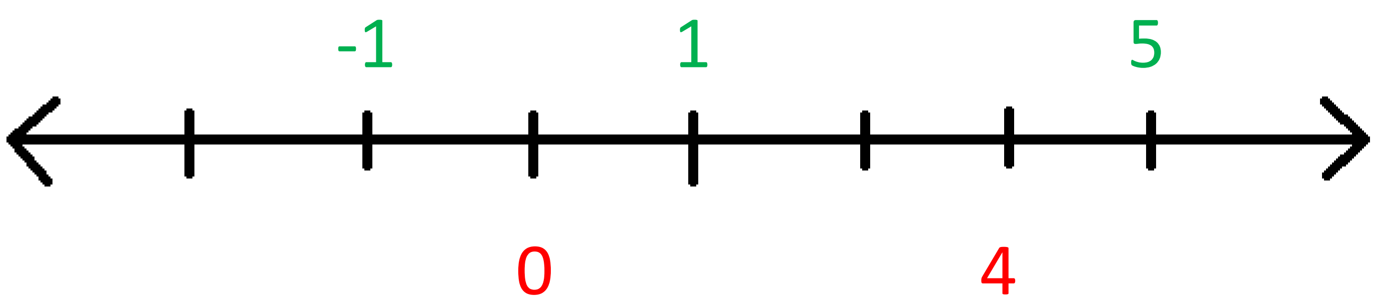
Derivative:

$$f'(x) = 3x^2 - 12x$$

Derivative is zero or undefined:

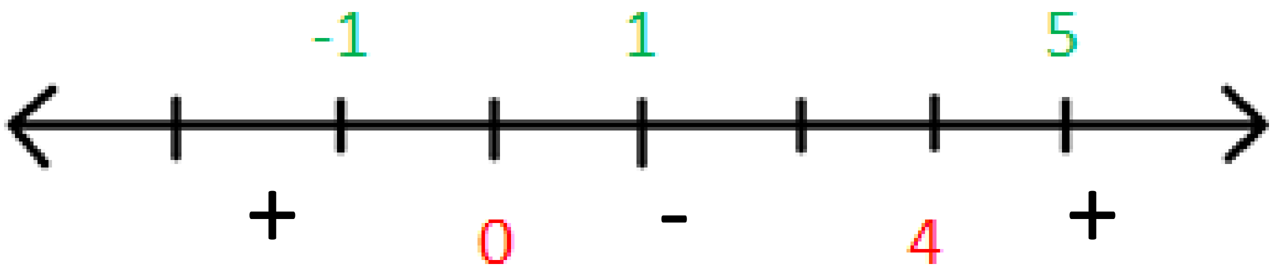
$$0 = 3x(x-4)$$

$$x = 0 \quad or \quad x = 4$$



Use test points in the derivative

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$



$$f'(-1) = 3(-1)(-1-4) = +\cdot - \cdot - = +$$
$$f'(1) = 3(1)(1-4) = +\cdot + \cdot - = -$$
$$f'(5) = 3(5)(5-4) = +++ = +$$

Conclusion:

Increasing on $(-\infty, 0) \cup (4, \infty)$

Decreasing on $(0,4)$

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

2. $f(x) = x^3 - 6x^2 + 15$

a) Critical Numbers: $x = 0,4$

b) Increasing: $(-\infty, 0) \cup (4, \infty)$

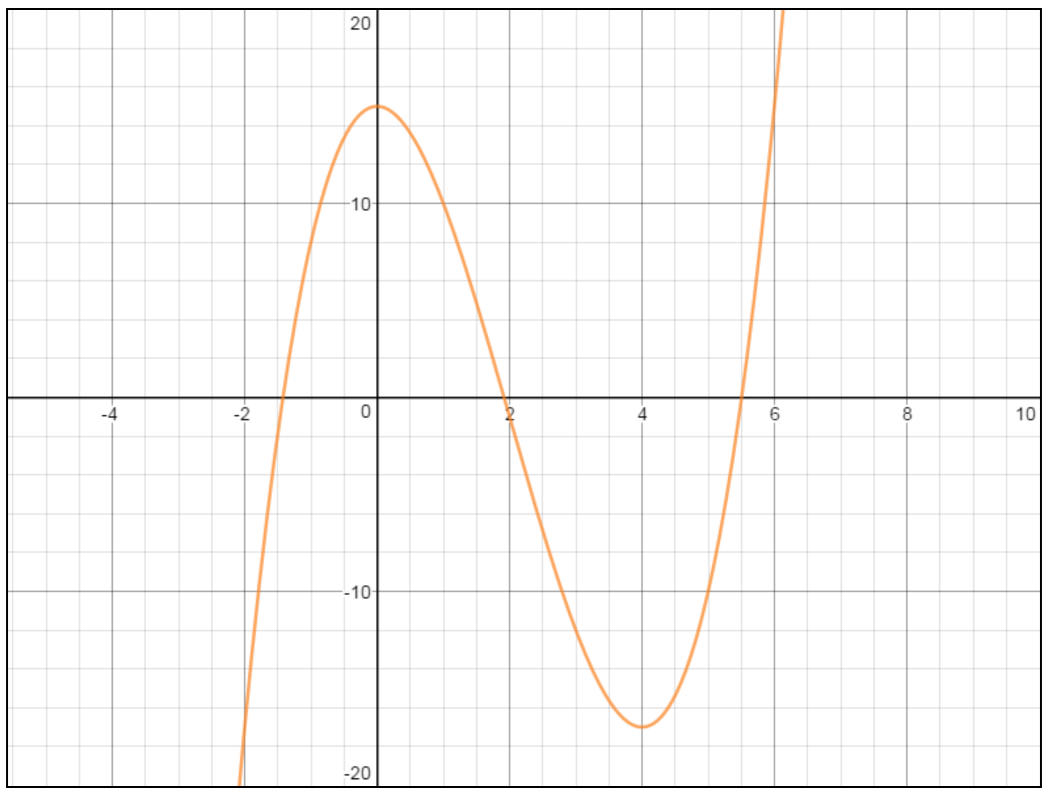
Decreasing: $(0,4)$

c) Changes increasing to decreasing at $x = 0$ so this is a maximum point: $\big(0, f(0)\big) = (0,15)$

Changes decreasing to increasing at $x = 4$ so this is a minimum point: $\big(4, f(4)\big) = (4, -17)$

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

2. $f(x) = x^3 - 6x^2 + 15$

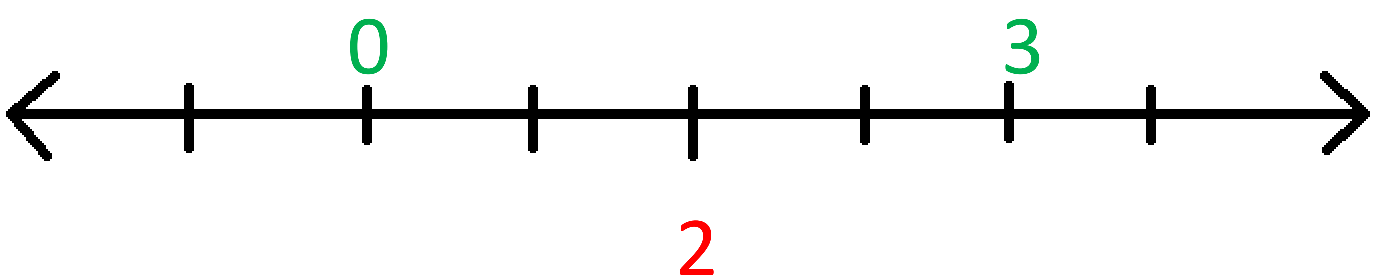


Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

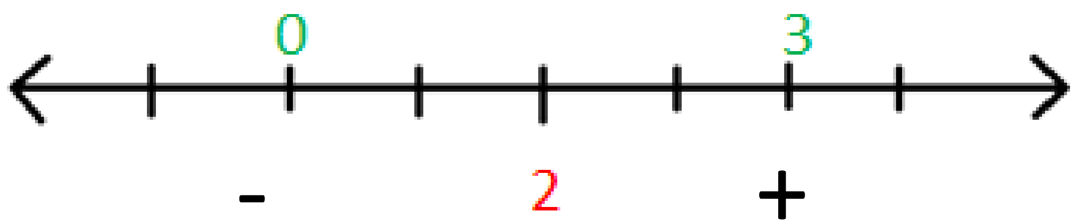
3. $f(x) = x^4 - 32x + 4$

$$f'(x) = 4x^3 - 32$$
$$0 = 4x^3 - 32$$
$$8 = x^3$$

Critical Number: $x = 2$



Using test points in the derivative $f'(x) = 4x^3 - 32$



$$f'(0) = -32 = negative$$
$$f'(3) = 4(3)^3 - 32 = positive$$

Conclusion:

Increasing: $(2, \infty)$

Decreasing: $(-\infty, 2)$

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

3. $f(x) = x^4 - 32x + 4$

Critical numbers: $x = 2$

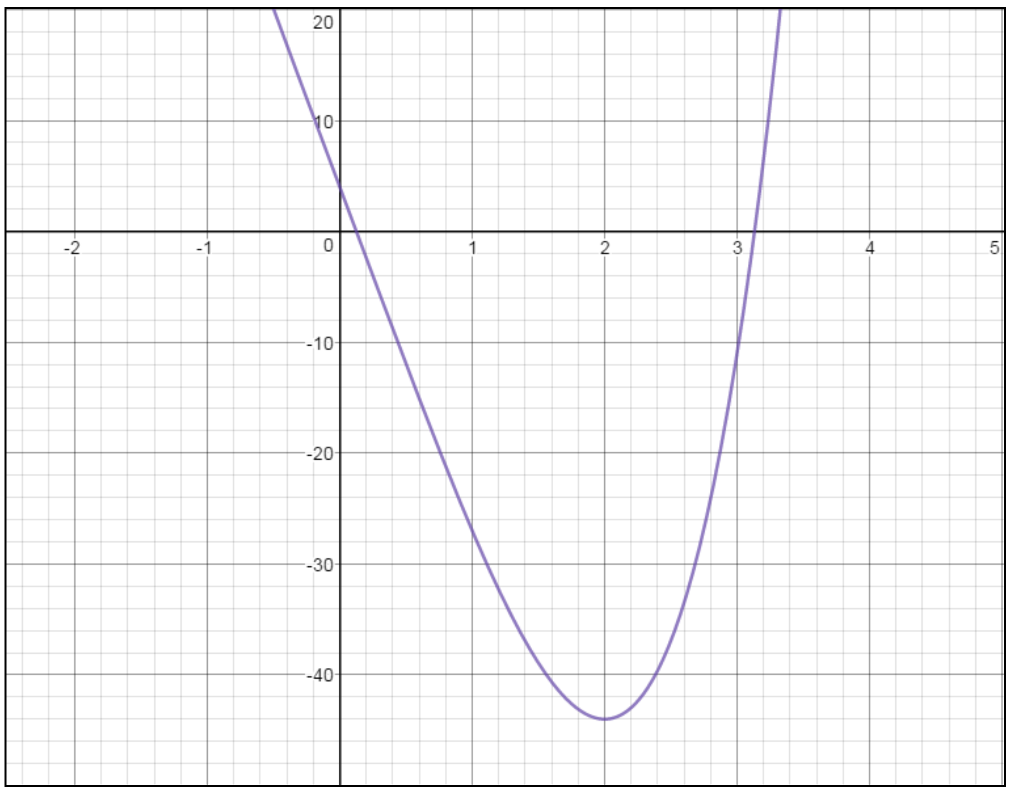
Increasing: $(2, \infty)$ Decreasing: $(-\infty, 2)$

Local minimum at $x = 2$ so the point $(2, f(2)) = (2, -44)$

- The derivative tells us where to look, the original function tells us what it is.

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

3. $f(x) = x^4 - 32x + 4$



Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

4. $f(x) = (x-3)^{1/3}$

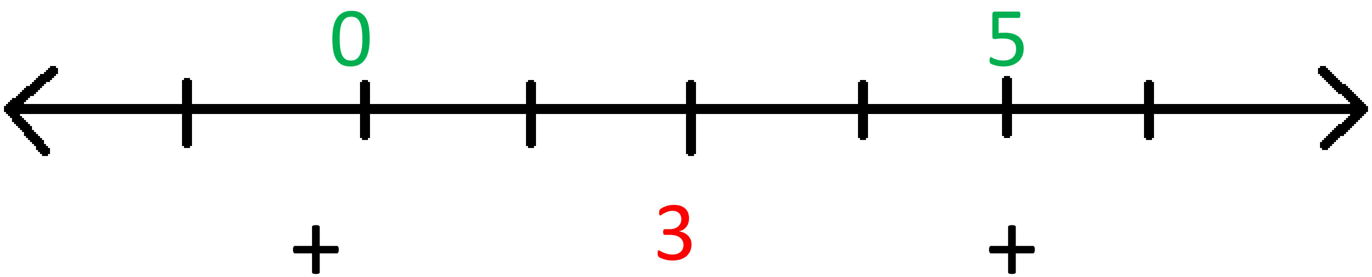
Derivative:

$$f'(x) = \frac{1}{3}(x-3)^{-\frac{2}{3}}(1)$$

$$f'(x) = \frac{1}{3\sqrt[3]{(x-3)^2}}$$

The derivative is never 0 as the numerator is never 0 but the derivative does not exist at $x = 3$.

Number line with derivative $f'(x) = \frac{1}{3\sqrt[3]{(x-3)^2}}$



$$f'(0) = \frac{1}{3\sqrt[3]{(0-3)^2}} = positive$$

$$f'(5) = \frac{1}{3\sqrt[3]{(5-3)^2}} = positive$$

Conclusion:

Increasing: $(-\infty, 3) \cup (3, \infty)$

Decreasing: never

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

4. $f(x) = (x - 3)^{1/3}$

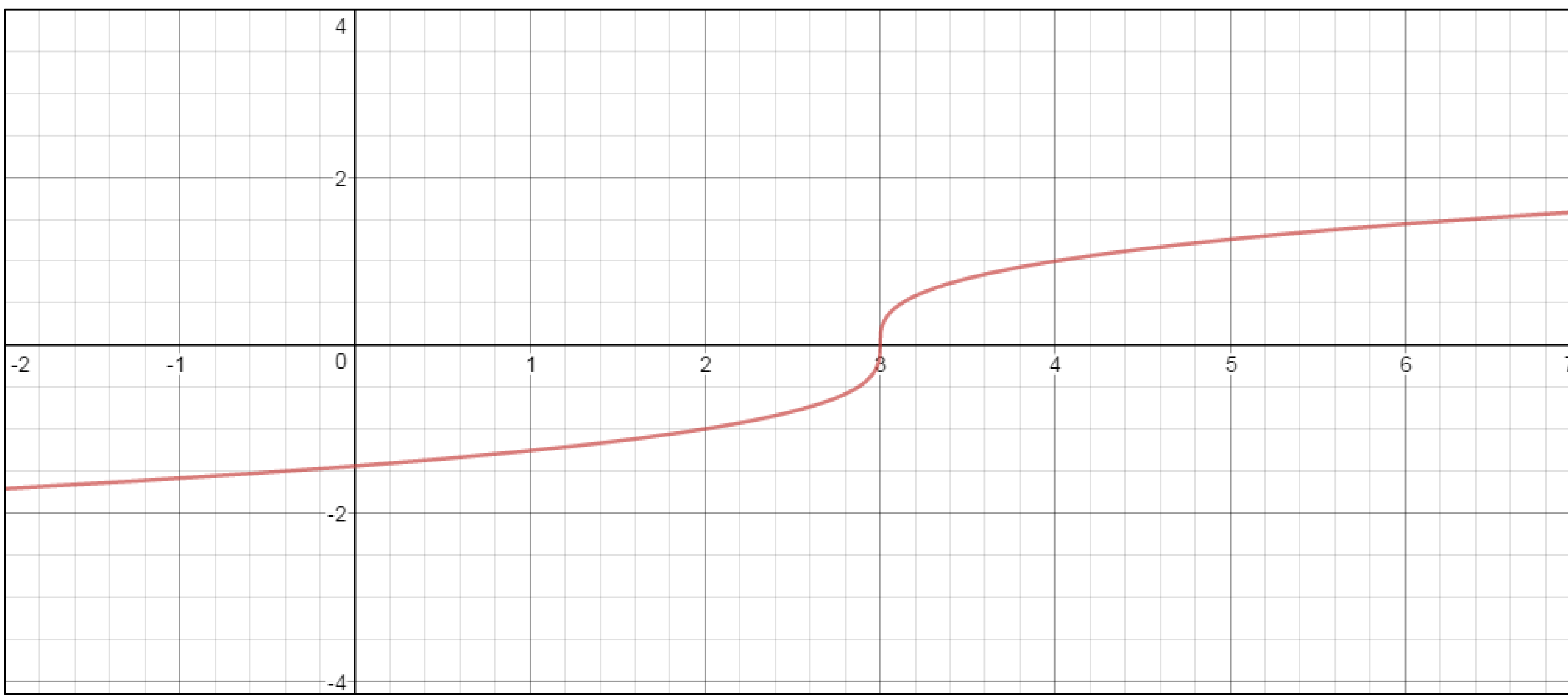
Critical number of x = 3

Increasing on $(-\infty, 3) \cup (3, \infty)$ and never decreasing.

Since the derivative never changes sign, there is no local minimum or maximum.

Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

4. $f(x) = (x - 3)^{1/3}$



Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

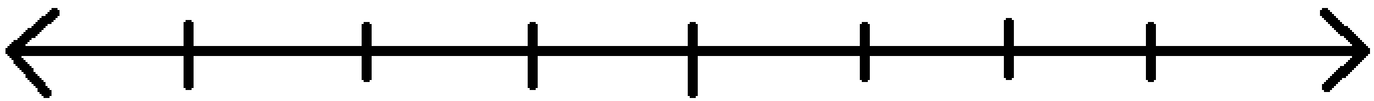
5. $f(x) = \frac{x}{x+3}$

Derivative:

$$f'(x) = \frac{(x+3)(1) - x(1)}{(x+3)^2} = \frac{3}{(x+3)^2}$$

Critical Number: $x = -3$

Test points in the derivative $f'(x) = \frac{3}{(x+3)^2}$



$$f'(0) = \frac{3}{(0+3)^2} = positive$$

Conclusion:

Increasing: $(-\infty, -3) \cup (-3, \infty)$

Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

5. $f(x) = \frac{x}{x+3}$

$$f'(x) = \frac{(x+3)(1) - x(1)}{(x+3)^2} = \frac{3}{(x+3)^2}$$

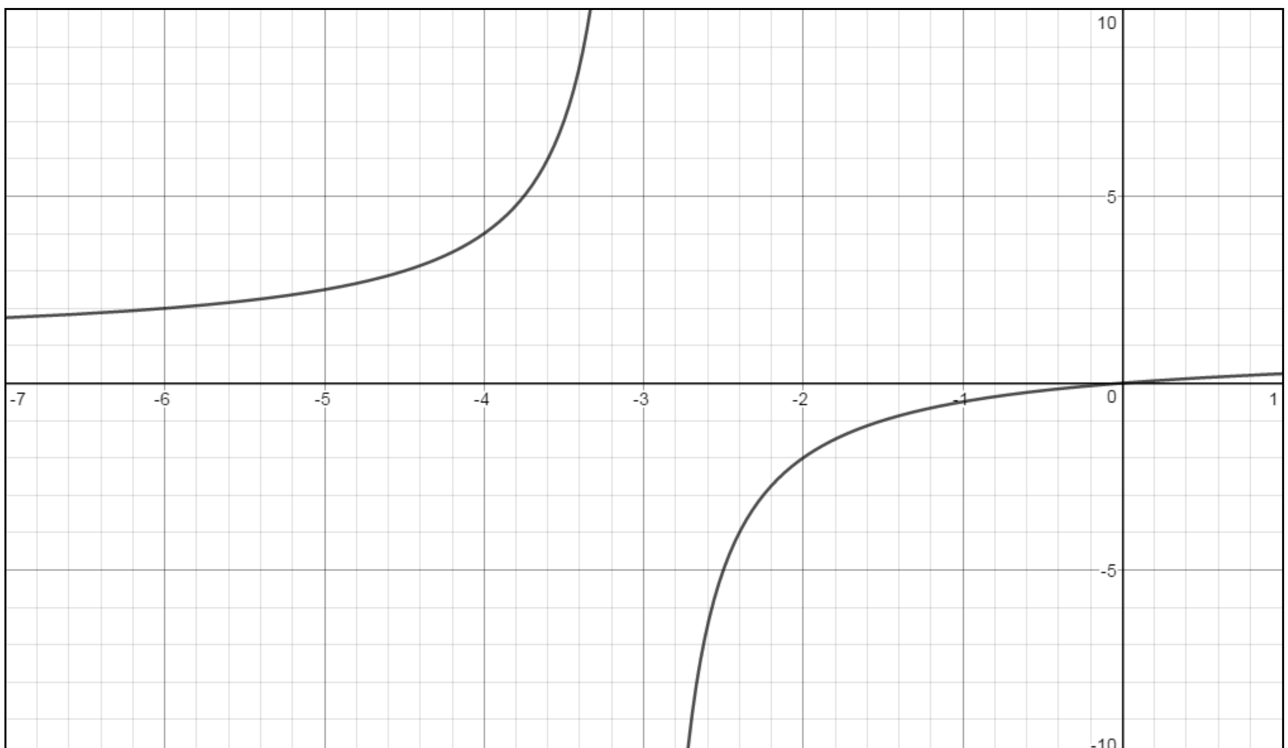
Critical Number: $x = -3$

Increasing: $(-\infty, -3) \cup (-3, \infty)$

No sign change in derivative so no extrema

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

5. $f(x) = \frac{x}{x+3}$



Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

6. $f(x) = \frac{x^2-3x-4}{x-2}$

Derivative:

$$f'(x) = \frac{(x-2)(2x-3)-(x^2-3x-4)(1)}{(x-2)^2}$$

$$f'(x) = \frac{2x^2-3x-4x+6-x^2+3x+4}{(x-2)^2}$$

$$f'(x) = \frac{x^2-4x+10}{(x-2)^2}$$

For $f'(x) = \frac{x^2-4x+10}{(x-2)^2}$, the only critical number is $x = 2$, where the denominator is zero. The numerator is never zero.

Test to the left of 2: $f'(0) = \frac{10}{positive} = pos$

Test to the right of 2: $f'(3) = \frac{7}{positive} = pos$

Conclusion:

Increasing: $(-\infty, 2) \cup (2, \infty)$

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

6. $f(x) = \frac{x^2-3x-4}{x-2}$

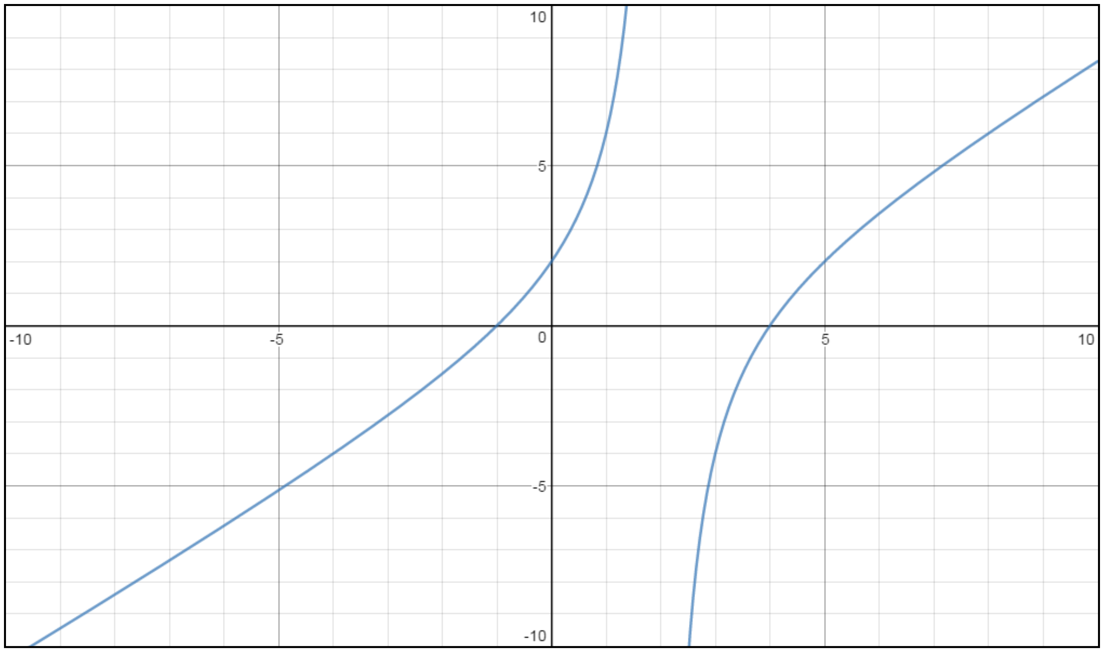
Critical Numbers: x = 2

Increasing: $(-\infty, 2) \cup (2, \infty)$

No change in sign of the derivative so no extrema.

Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

6. $f(x) = \frac{x^2-3x-4}{x-2}$



Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

7. $f(x) = \sin x \cos x + 5$ on $(0,2\pi)$

Derivative:

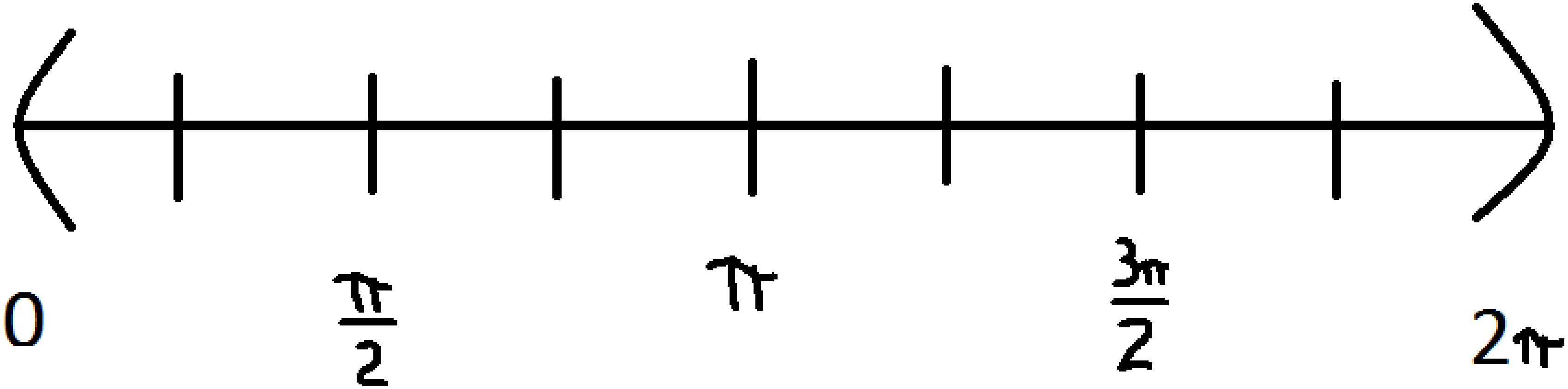
$$f'(x) = \sin x\,(-\sin x) + \cos x\,(\cos x)$$
$$f'(x) = cos^2 x - sin^2 x$$

Critical Numbers:

$$0 = cos^2 x - sin^2 x$$
$$sin^2 x = cos^2 x$$

Which occurs at $\frac{\pi}{4}$ in all four quadrants.

Test points in $f'(x) = cos^2x - sin^2x$



$$f'\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad f'\left(\frac{\pi}{2}\right) = -1$$

$$f'(\pi) = 1 \qquad f'\left(\frac{3\pi}{2}\right) = -1$$

$$f'\left(\frac{11\pi}{6}\right) = \frac{1}{2}$$

Increasing: $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing: $(\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})$

Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

7. $f(x) = \sin x \cos x + 5$ on $(0, 2\pi)$

Critical Numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Increasing: $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$

Intervals only involving x's

Decreasing: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

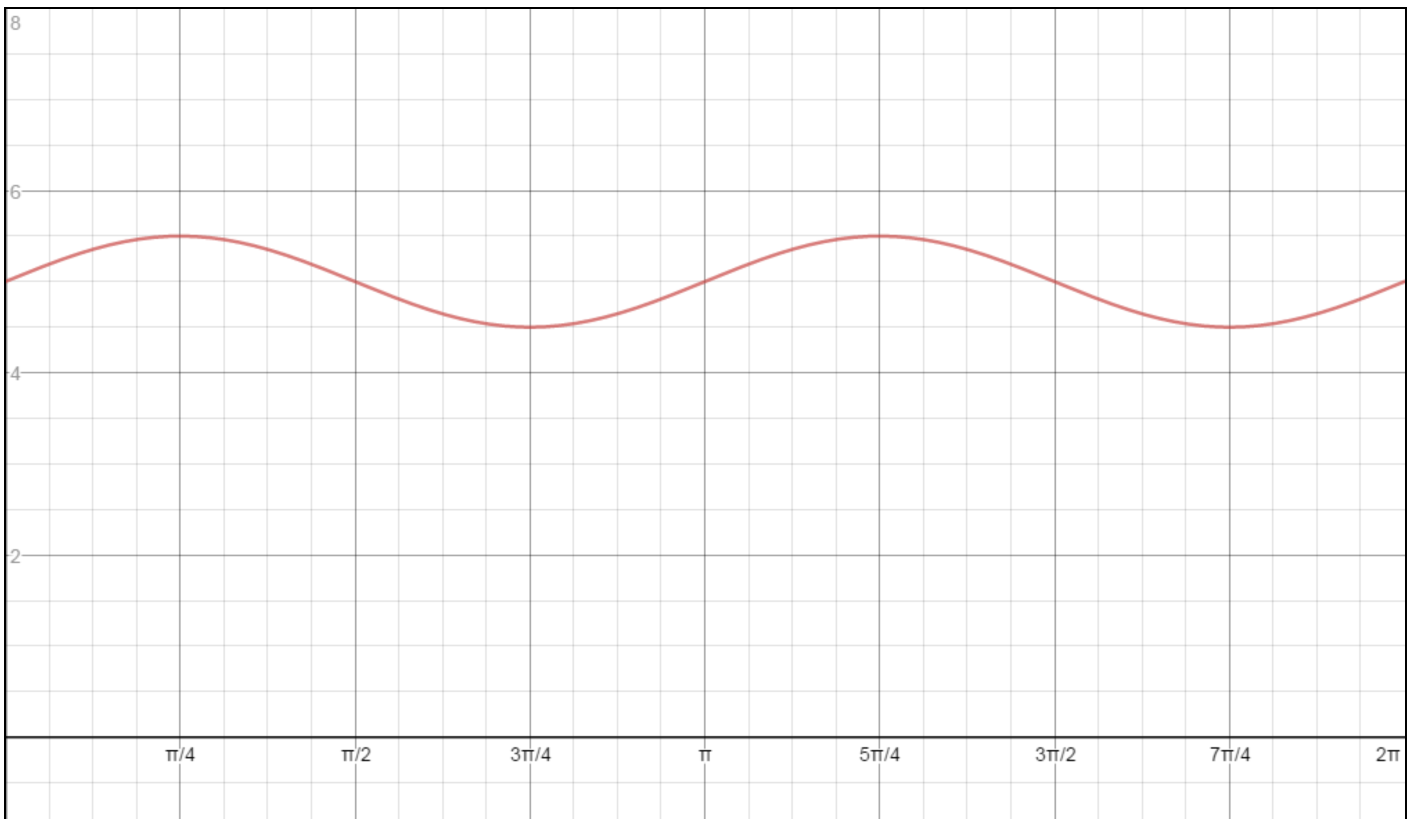
Maximums at: $\left(\frac{\pi}{4}, \frac{11}{2}\right)$ and $\left(\frac{5\pi}{4}, \frac{11}{2}\right)$

Minimums at: $\left(\frac{3\pi}{4}, \frac{9}{2}\right)$ and $\left(\frac{7\pi}{4}, \frac{9}{2}\right)$

Points of the form (x,y)

Examples: (a) Find the critical numbers of f, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the FDT to identify all relative extrema.

7. $f(x) = \sin x \cos x + 5$ on $(0, 2\pi)$



Coughing forces the trachea (windpipe) to contract, which affects the velocity v of the air passing through the trachea. The velocity of the air during coughing is $v = k(R-r)r^2$ for $\mathbf{0} \leq r < R$, where k is a constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce maximum air velocity?

- Find r when $v' = 0$

$$\frac{dv}{dr} = k[(R-r)2r + r^2(-1)]$$
$$= k[2Rr - 2r^2 - r^2]$$
$$= k[2Rr - 3r^2]$$

- The derivative is zero when $0 = r(2R - 3r)$ or when $r = 0$ and $r = \frac{2R}{r}$

- The critical number of $r = 0$ will not produce maximum air flow as the radius of the trachea at zero will produce NO air flow. This is a bad situation!
- The critical number $r = \frac{2R}{3}$ is a radius that is 2/3 the normal radius of your trachea in order to provide maximum velocity of a cough.

- Good luck doing that on purpose!!!

End of Lecture