Concavity and the Second Derivative Test

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Suggested Review Topics

- Algebra skills reviews suggested:
  - None

- Trigonometric skills reviews suggested:
  - None
Applications of Differentiation

Concavity and the Second Derivative Test
Definition of Concavity

- Let $f$ be differentiable on an open interval $I$. The graph of $f$ is concave upward on $I$ if $f'$ is increasing on the interval and concave downward on $I$ if $f'$ is decreasing on the interval.

- **NOTE:** This is talking about the derivative increasing or decreasing...

- To talk about the derivative increasing or decreasing we need to be able to find the second derivative, $f''$. 
Test for Concavity

- Let $f$ be a function whose second derivative exists on an open interval $I$.
  1. If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward on $I$.
  2. If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward on $I$. 
Definition of Point of Inflection

• Let $f$ be a function that is continuous on an open interval and let $c$ be a point in the interval. If the graph of $f$ has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of $f$ if the concavity of $f$ changes from upward to downward (or downward to upward) at the point.
Theorem 3.8

If \((c, f(c))\) is a point of inflection of the graph of \(f\), then either \(f''(x) = 0\) or \(f''\) does not exist at \(x = c\).
Examples: Determine the open intervals on which the graph is concave upward or concave downward.

1. \( f(x) = -x^3 + 3x^2 - 2 \)

First derivative: \( f'(x) = -3x^2 + 6x \)

Second derivative: \( f''(x) = -6x + 6 \)

Solve \( f''(x) = 0 \):

\[
0 = -6x + 6
\]

\[
x = 1
\]

Use the number line approach like we did for critical numbers.
Using \( x = 1 \) and \( f''(x) = -6x + 6 \)

\[
f''(0) = -6(0) + 6 = \text{positive}
\]

\[
f''(2) = -6(2) + 6 = \text{negative}
\]

Concave Up: \( (-\infty, 1) \)

Concave Down: \( (1, \infty) \)
Examples: Determine the open intervals on which the graph is concave upward or concave downward.

2. $y = x^5 - 5x + 2$
First derivative: $y' = 5x^4 - 5$
Second derivative: $y'' = 20x^3$
We know the second derivative is zero when $x$ is zero, positive when $x$ is positive, and negative when $x$ is negative.
Concave up: $(0, \infty)$  Concave down: $(-\infty, 0)$
Examples: Determine the open intervals on which the graph is concave upward or concave downward.

3. \( f(x) = \frac{x^2}{x^2+1} \)

First derivative:

\[
f'(x) = \frac{(x^2 + 1)2x - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}
\]

Second derivative:

\[
f''(x) = \frac{(x^2 + 1)^2(2) - 2x(2(x^2 + 1)(2x))}{(x^2 + 1)^4}
\]
\[ f''(x) = \frac{(x^2 + 1)^2(2) - 2x(2(x^2 + 1)(2x))}{(x^2 + 1)^4} \]
\[ = \frac{2(x^2 + 1) - 8x^2}{(x^2 + 1)^3} = \frac{2 - 6x^2}{(x^2 + 1)^3} \]

- Denominator is always positive.
- Numerator: \( 0 = 2 - 6x^2 \rightarrow 6x^2 = 2 \) which becomes \( x^2 = \frac{1}{3} \rightarrow x = \pm \sqrt{1/3} = \pm \frac{\sqrt{3}}{3} \)
Using $f''(x) = \frac{2-6x^2}{(x^2+1)^3}$

$f''(-5) = \frac{2-6(-5)^2}{\text{positive}} = \frac{\text{neg}}{\text{pos}} = \text{neg}$

$f''(0) = \frac{2}{\text{pos}} = \text{pos}$

$f''(5) = \frac{2-6(5)^2}{\text{positive}} = \text{neg}$

Concave Up: $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

Concave Down: $\left(-\infty, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \infty\right)$
Second Derivative Test

• Let $f$ be a function such that $f'(c) = 0$ and the second derivative of $f$ exists on an open interval containing $c$.

  1. If $f''(c) > 0$, then $f$ has a relative minimum at $(c, f(c))$.
  2. If $f''(c) < 0$, then $f$ has a relative maximum at $(c, f(c))$.
  3. If $f''(c) = 0$ the test fails. That is, $f$ may have a relative maximum, a relative minimum, or neither. In such cases revert to the first derivative test.
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

1. \( f(x) = -x^4 + 24x^2 \)

First derivative:

\[
f'(x) = -4x^3 + 48x
\]

First derivative will give us critical numbers, increasing and decreasing, and extrema.

Solve:

\[
0 = -4x(x^2 - 12)
\]

\[
x = 0 \quad \text{and} \quad x = \pm 2\sqrt{3}
\]
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

1. \( f(x) = -x^4 + 24x^2 \)

First derivative:

\[
f'(x) = -4x^3 + 48x
\]

Second derivative:

\[
f''(x) = -12x^2 + 48
\]

Second derivative gives us location of points of inflection, concavity, and extrema classifications.

\[
0 = -12x^2 + 48
\]
\[
x^2 = 4 \rightarrow x = \pm 2
\]
Derivatives: \( f'(x) = -4x^3 + 48x \) and 
\( f''(x) = -12x^2 + 48 \)

\[
\begin{align*}
  f'(-4) &= 64 \\
  f'(1) &= 44 \\
  f''(-4) &= \text{neg} \\
  f''(4) &= \text{neg} \\
  f'(-1) &= -44 \\
  f'(4) &= -64 \\
  f''(1) &= \text{pos} \\
  f''(4) &= \text{neg}
\end{align*}
\]
Putting this on the number line we have:

Inc: $(-\infty, -2\sqrt{3}), (0, 2\sqrt{3})$

Dec: $(-2\sqrt{3}, 0), (2\sqrt{3}, \infty)$

CC Up: $(-2, 2)$  CC Down: $(-\infty, -2), (2, \infty)$

Max Points: $(\pm 2\sqrt{3}, 144)$ Min Point: $(0,0)$

Points of Inflection: $(\pm 2, 80)$
The graph of \( f(x) = -x^4 + 24x^2 \)
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

2. \( f(x) = -(x - 5)^2 \)

1\(^{st}\) derivative: \( f'(x) = -2(x - 5) = -2x + 10 \)

2\(^{nd}\) derivative: \( f''(x) = -2 \)

Critical number \( x = 5 \), no points of inflection

Inc: \(( -\infty, 5 )\)  Dec: \(( 5, \infty )\)  Max: \(( 5, 0 )\)

CC Up: never  CC down: \(( -\infty, \infty )\)
The graph of $f(x) = -(x - 5)^2$
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

3. \( f(x) = x^3 - 5x^2 + 7x \)

First derivative:
\[
f'(x) = 3x^2 - 10x + 7 = (3x - 7)(x - 1)
\]

Second derivative: \( f''(x) = 6x - 10 \)

Critical Numbers: \( x = \frac{7}{3}, 1 \) (possible max/min)

Zeros of 2\(^{nd}\): \( x = \frac{5}{3} \) (possible point of inflection)
Derivatives: \( f'(x) = 3x^2 - 10x + 7 \) and \( f''(x) = 6x - 10 \)

\[ f'(0) = 7 \quad f'(2) = -1 \quad f'(5) = 32 \]

\[ f''(0) = -10 \quad f''(2) = 2 \]

Inc: \((-\infty, 1), (\frac{7}{3}, \infty)\) Dec: \((1, \frac{7}{3})\)

CC Up: \((\frac{5}{3}, \infty)\) CC down: \((-\infty, \frac{5}{3})\)

Max: \((1, 3)\) Min: \((\frac{7}{3}, \frac{49}{27})\) POI: \((\frac{5}{3}, \frac{65}{27})\)
The graph of \( f(x) = x^3 - 5x^2 + 7x \)
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

4. \( g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2 \)

First derivative

\[
g'(x) = -\frac{1}{8} \left[ (x + 2)^2 \cdot 2(x - 4) + (x - 4)^2 \cdot 2(x + 2) \right] \\
= -\frac{1}{8} \left[ 2(x + 2)(x - 4)(x + 2 + x - 4) \right] \\
= -\frac{1}{4} (x + 2)(x - 4)(2x - 2) \\
= -\frac{1}{2} (x + 2)(x - 4)(x - 1) \]
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

4. \( g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2 \)

Second derivative:

\[
g''(x) = -\frac{1}{2}[(x + 2)(x - 4) + (x + 2)(x - 1) + (x - 4)(x - 1)]
\]

\[
= -\frac{1}{2}[x^2 - 2x - 8 + x^2 + x - 2 + x^2 - 5x + 4]
\]

\[
= -\frac{1}{2}[3x^2 - 6x - 6] = -\frac{3}{2}(x^2 - 2x - 2)
\]
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

4. \( g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2 \)

First: \( g'(x) = -\frac{1}{2}(x + 2)(x - 4)(x - 1) \)

Second: \( g''(x) = -\frac{3}{2}(x^2 - 2x - 2) \)

CN: \( x = -2, 4, 1 \) Possible POI: \( x = 1 \pm \sqrt{3} \)
Function: \( g(x) = -\frac{1}{8} (x + 2)^2 (x - 4)^2 \)

First: \( g'(x) = -\frac{1}{2} (x + 2)(x - 4)(x - 1) \)

Second: \( g''(x) = -\frac{3}{2} (x^2 - 2x - 2) \)

CN: \( x = -2, 4, 1 \quad \text{Possible POI: } x = 1 \pm \sqrt{3} \)
Function: \( g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2 \)

First: \( g'(x) = -\frac{1}{2}(x + 2)(x - 4)(x - 1) \)

Second: \( g''(x) = -\frac{3}{2}(x^2 - 2x - 2) \)

\[
\begin{array}{c}
g'(-4) = + \quad g'(0) = - \quad g'(2) = + \quad g'(10) = + \\
g''(-4) = - \quad g''(0) = + \quad g''(10) = + \\
\end{array}
\]

Inc: \((-\infty, -2), (1, 4)\)  Dec: \((-2, 1), (4, \infty)\)

CC Up: \((1 - \sqrt{3}, 1 + \sqrt{3})\)

CC Down: \((-\infty, 1 - \sqrt{3}), (1 + \sqrt{3}, \infty)\)
Function: $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

First: $g'(x) = -\frac{1}{2}(x + 2)(x - 4)(x - 1)$

Second: $g''(x) = -\frac{3}{2}(x^2 - 2x - 2)$

Maximum Points: $(-2,0)$ and $(4,0)$

Minimum Point: $(1, -\frac{81}{8})$

Points of Inflection: $(1 - \sqrt{3}, -\frac{9}{2})$ and $(1 + \sqrt{3}, -\frac{9}{2})$
The graph of \( g(x) = -\frac{1}{8} (x + 2)^2 (x - 4)^2 \)
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

5. \( y = \frac{x}{x-1} \)

1\(^{\text{st}}\) derivative: \( y' = \frac{(x-1)(1)-x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2} \)

2\(^{\text{nd}}\) derivative:
\[
y'' = \frac{(x-1)^2(0) - (-1)2(x-1)(1)}{(x-1)^4} = \frac{2}{(x-1)^3}
\]

The only critical number for either derivative occurs at \( x = 1 \). Notice this value is not in the domain of the original function either.
Using $y' = \frac{-1}{(x-1)^2}$ and $y'' = \frac{2}{(x-1)^3}$ and $x = 1$:

\[ y'(0) = \frac{\text{neg}}{\text{pos}} \quad y'(2) = \frac{\text{neg}}{\text{pos}} \quad \text{Dec: } (-\infty, 1), (1, \infty) \]

\[ y''(0) = \frac{\text{pos}}{\text{neg}} \quad y''(2) = \frac{\text{pos}}{\text{pos}} \]

CC Up: $(1, \infty)$  \quad CC down: $(-\infty, 1)$

No extrema as $x = 1$ is not in the domain of the function.
The graph of $y = \frac{x}{x-1}$
End of Lecture