Concavity and the Second Derivative Test

By Tuesday J. Johnson

Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

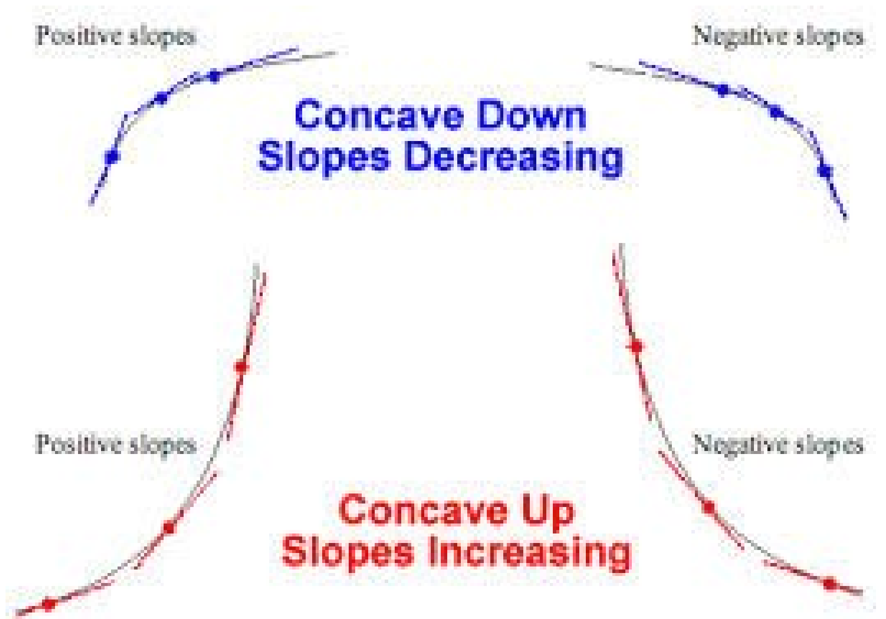
Concavity and the Second Derivative Test

Definition of Concavity

- Let f be differentiable on an open interval I. The graph of f is concave upward on I if f' is increasing on the interval and concave downward on I if f' is decreasing on the interval.
- NOTE: This is talking about the derivative increasing or decreasing…
- To talk about the derivative increasing or decreasing we need to be able to find the second derivative, f''.

Test for Concavity

- Let *f* be a function whose second derivative exists on an open interval *I*.
  1. If $f''(x) > 0$ for all *x* in *I*, then the graph of *f* is concave upward on *I.*
  2. If $f''(x) < 0$ for all *x* in *I*, then the graph of *f* is concave downward on *I*

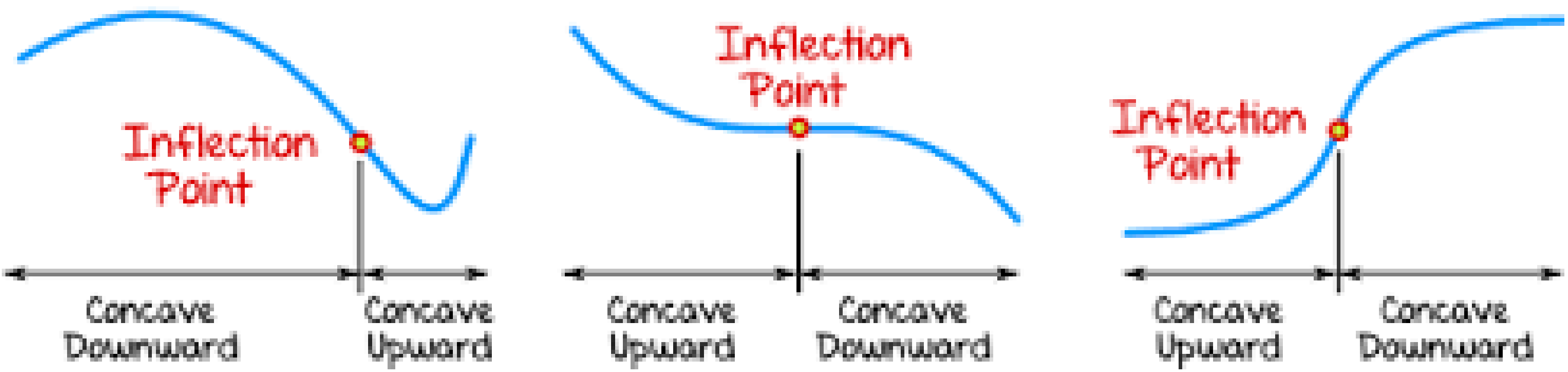


Definition of Point of Inflection

- Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $\left(c, f(c)\right)$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(x) = 0$ or f'' does not exist at $x = c$.



Examples: Determine the open intervals on which the graph is concave upward or concave downward.

1. $f(x) = -x^3 + 3x^2 - 2$

First derivative: $f'(x) = -3x^2 + 6x$

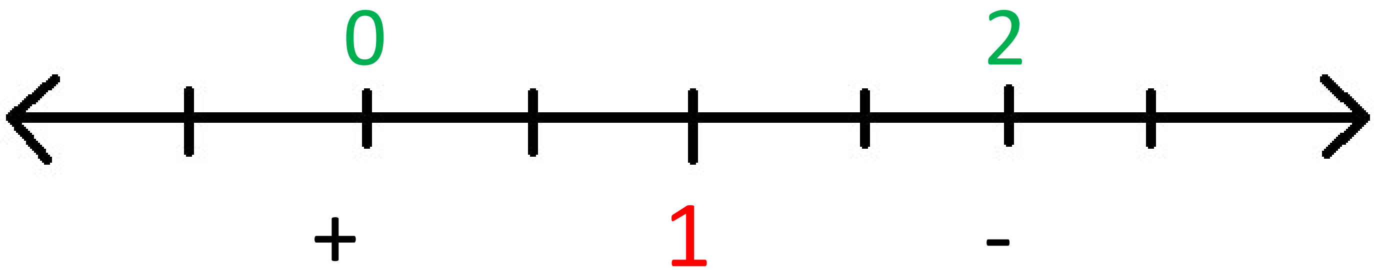
Second derivative: $f''(x) = -6x + 6$

Solve $f''(x) = 0$:

$$0 = -6x + 6$$
$$x = 1$$

Use the number line approach like we did for critical numbers.

Using $x = 1$ and $f''(x) = -6x + 6$



$$f''(0) = -6(0) + 6 = positive$$
$$f''(2) = -6(2) + 6 = negative$$

Concave Up: $(-\infty, 1)$

Concave Down: $(1, \infty)$

Examples: Determine the open intervals on which the graph is concave upward or concave downward.

2. $y = x^5 - 5x + 2$

First derivative: $y' = 5x^4 - 5$

Second derivative: $y'' = 20x^3$

We know the second derivative is zero when x is zero, positive when x is positive, and negative when x is negative.

Concave up: $(0, \infty)$ Concave down: $(-\infty, 0)$

Examples: Determine the open intervals on which the graph is concave upward or concave downward.

3. $f(x) = \frac{x^2}{x^2+1}$

First derivative:

$$f'(x) = \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

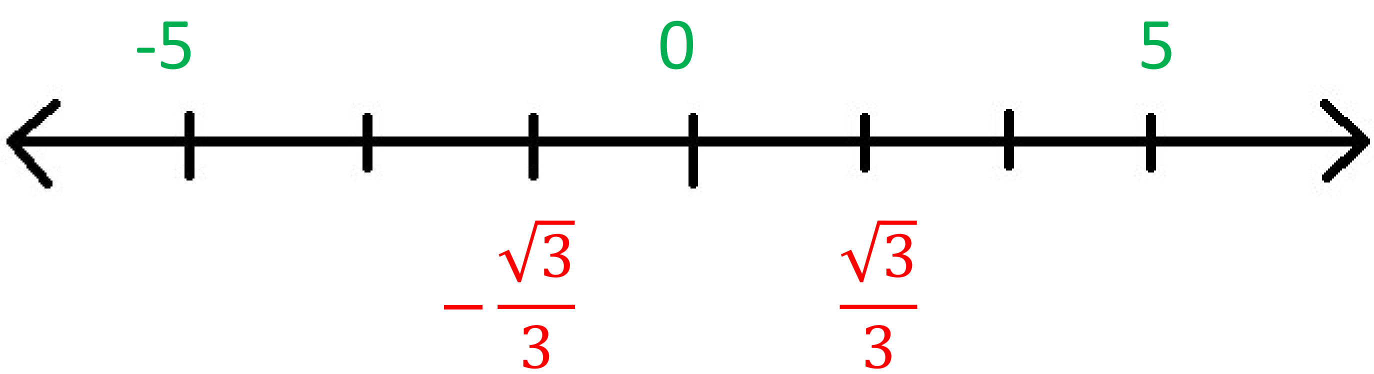
Second derivative:

$$f''(x) = \frac{(x^2+1)^2(2) - 2x(2(x^2+1)(2x)}{(x^2+1)^4}$$

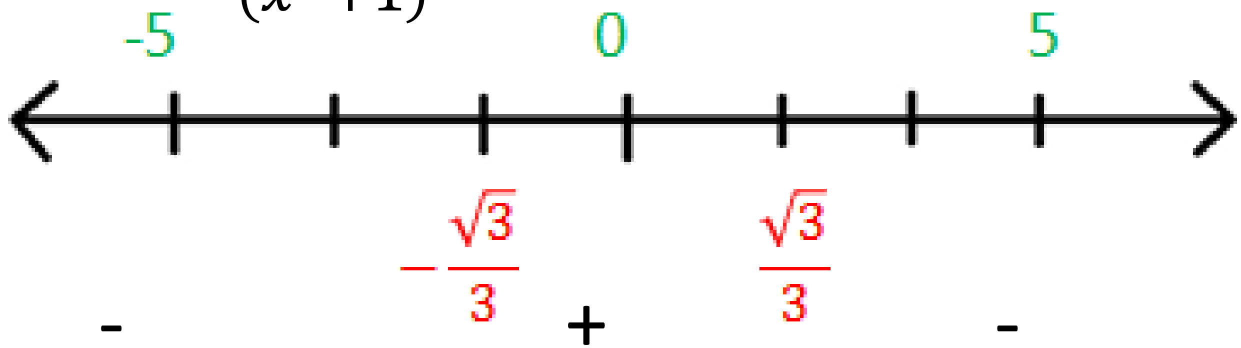
$$f''(x) = \frac{(x^2+1)^2(2) - 2x(2(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{2(x^2+1) - 8x^2}{(x^2+1)^3} = \frac{2-6x^2}{(x^2+1)^3}$$

- Denominator is always positive.
- Numerator: $0 = 2 - 6x^2 \rightarrow 6x^2 = 2$ which becomes $x^2 = \frac{1}{3} \rightarrow x = \pm\sqrt{1/3} = \pm\frac{\sqrt{3}}{3}$



Using $f''(x) = \frac{2-6x^2}{(x^2+1)^3}$



$$f''(-5) = \frac{2-6(-5)^2}{positive} = \frac{neg}{pos} = neg$$

$$f''(0) = \frac{2}{pos} = pos$$

$$f''(5) = \frac{2-6(5)^2}{positive} = neg$$

Concave Up: $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

Concave Down: $\left(-\infty, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \infty\right)$

Second Derivative Test

- Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.
  1. If $f''(c) > 0$, then f has a relative minimum at $\left(c, f(c)\right)$.
  2. If $f''(c) < 0$, then f has a relative maximum at $\left(c, f(c)\right)$.
  3. If $f''(c) = 0$ the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases revert to the first derivative test.

Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

1. $f(x) = -x^4 + 24x^2$

First derivative:

$$f'(x) = -4x^3 + 48x$$

First derivative will give us critical numbers, increasing and decreasing, and extrema.

Solve: $$0 = -4x(x^2 - 12)$$

$$x = 0 \quad and \quad x = \pm 2\sqrt{3}$$

Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

1. $f(x) = -x^4 + 24x^2$

First derivative:

$$f'(x) = -4x^3 + 48x$$

Second derivative:

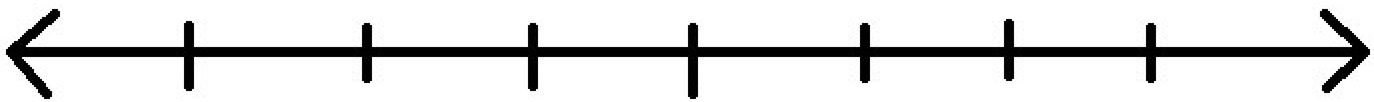
$$f"(x) = -12x^2 + 48$$

Second derivative gives us location of points of inflection, concavity, and extrema classifications.

$$0 = -12x^2 + 48$$

$$x^2 = 4 \rightarrow x = \pm 2$$

Derivatives: $f'(x) = -4x^3 + 48x$ and $f''(x) = -12x^2 + 48$



$f'(-4) = 64$

$f'(1) = 44$

$f''(-4) = neg$

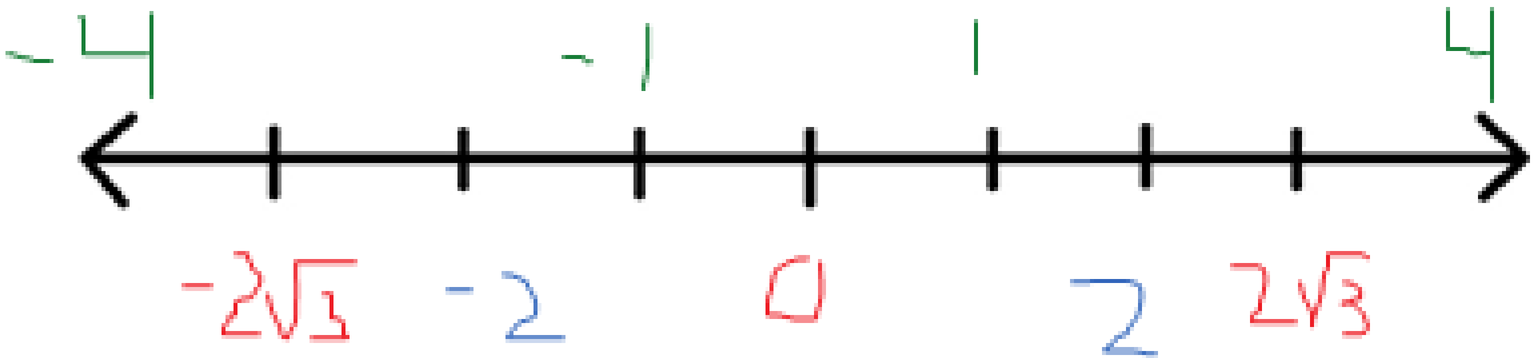
$f''(4) = neg$

$f'(-1) = -44$

$f'(4) = -64$

$f''(1) = pos$

Putting this on the number line we have:



Inc: $\left(-\infty,-2\sqrt{3}\right),\left(0,2\sqrt{3}\right)$

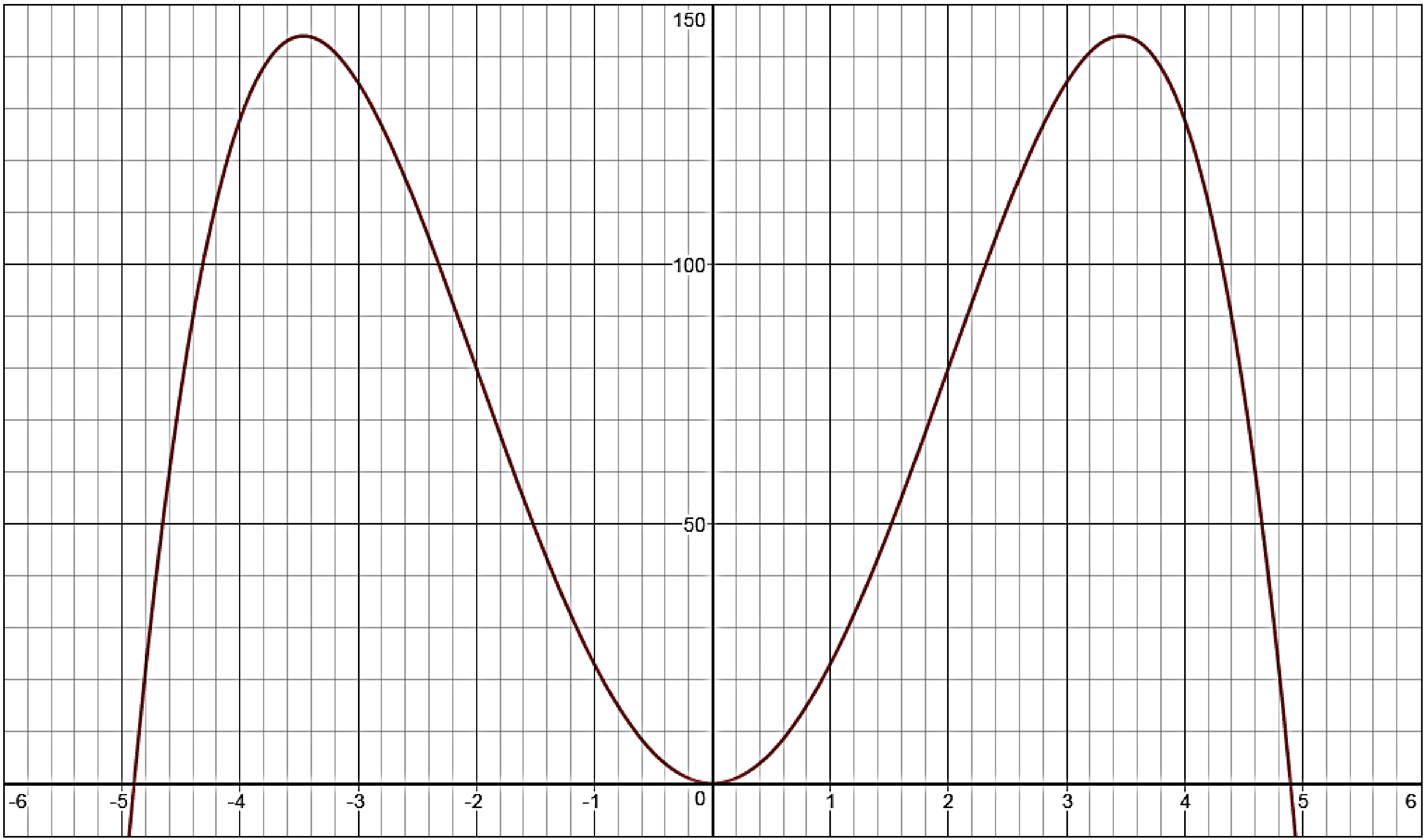
Dec: $\left(-2\sqrt{3},0\right),(2\sqrt{3},\infty)$

CC Up: $(-2,2)$ CC Down: $(-\infty,-2),(2,\infty)$

Max Points: $(\pm 2\sqrt{3},144)$ Min Point: $(0,0)$

Points of Inflection: $(\pm 2,80)$

The graph of $f(x) = -x^4 + 24x^2$



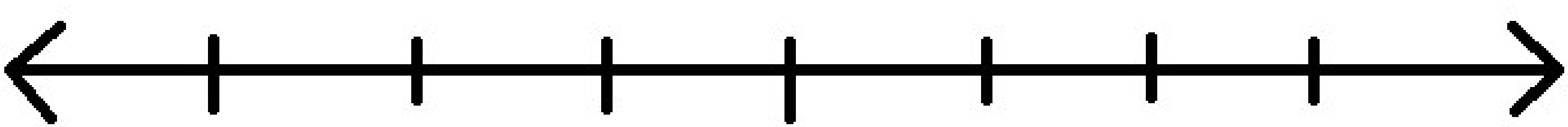
Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

2. $f(x) = -(x-5)^2$

1st derivative: $f'(x) = -2(x-5) = -2x + 10$

2nd derivative: $f''(x) = -2$

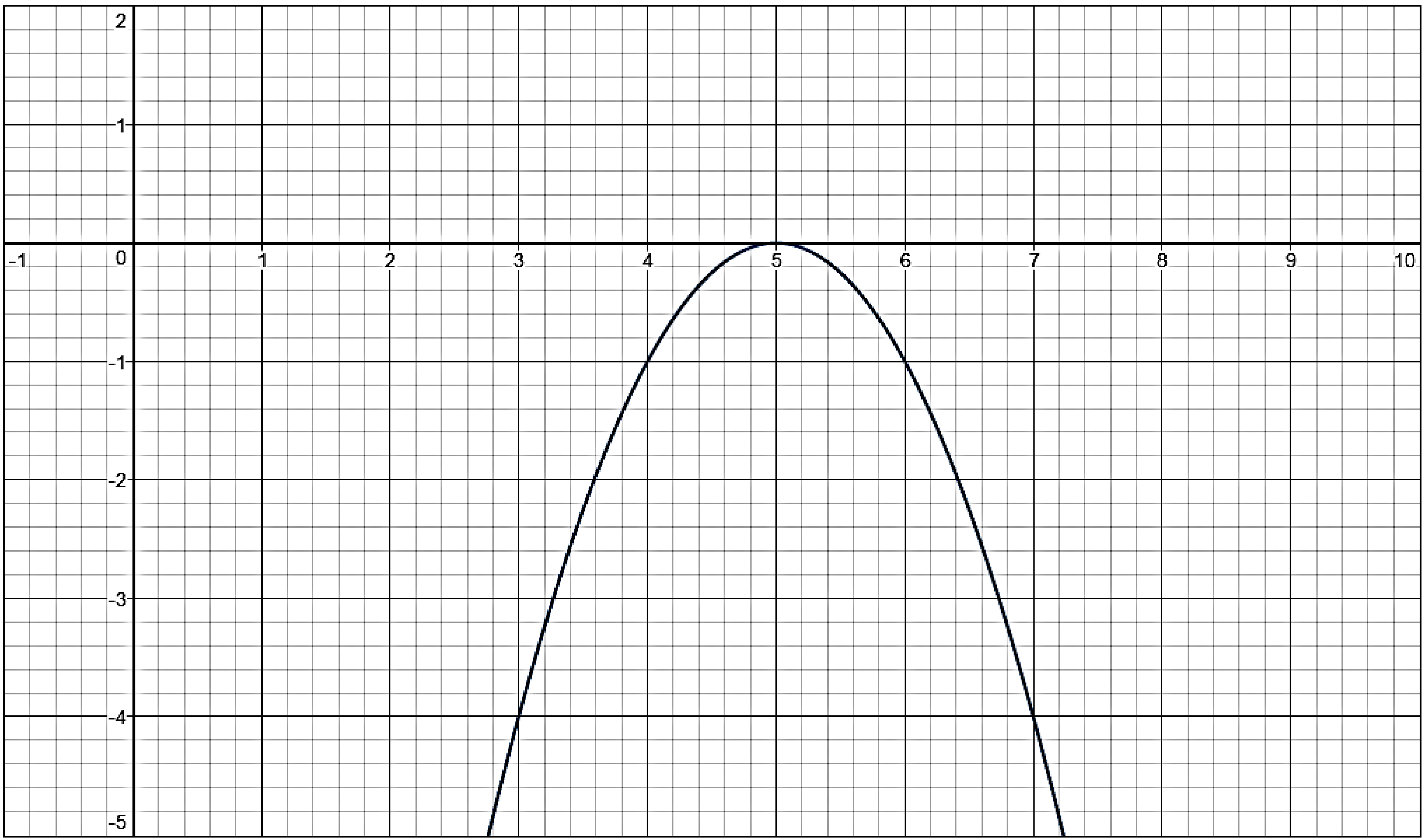
Critical number $x = 5$, no points of inflection



Inc: $(-\infty, 5)$ Dec: $(5, \infty)$ Max: $(5,0)$

CC Up: never CC down: $(-\infty, \infty)$

The graph of $f(x) = -(x-5)^2$



Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

3. $f(x) = x^3 - 5x^2 + 7x$

First derivative:

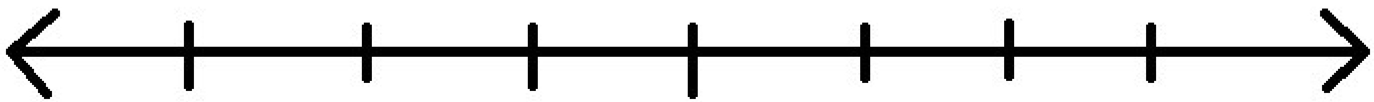
$$f'(x) = 3x^2 - 10x + 7 = (3x - 7)(x - 1)$$

Second derivative: $f''(x) = 6x - 10$

Critical Numbers: $x = \frac{7}{3}, 1$ (possible max/min)

Zeros of 2nd: $x = \frac{5}{3}$ (possible point of inflection)

Derivatives: $f'(x) = 3x^2 - 10x + 7$ and $f"(x) = 6x - 10$



$f'(0) = 7$ $f'(2) = -1$ $f'(5) = 32$

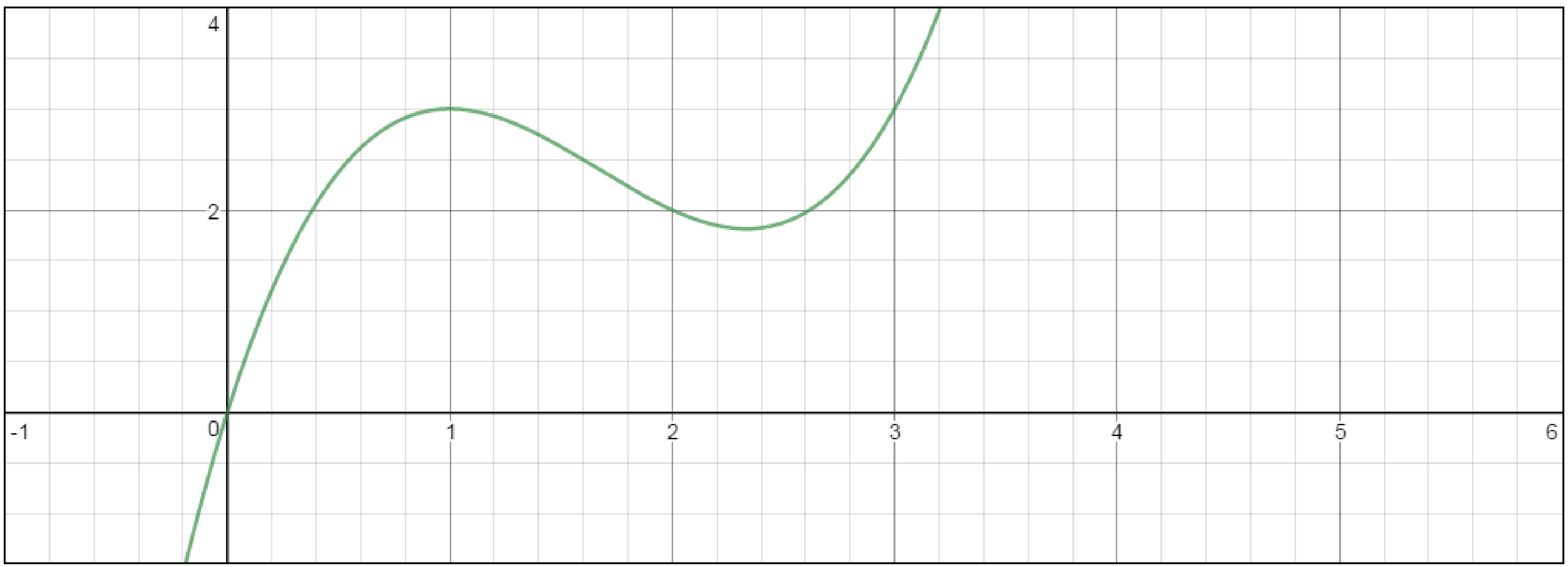
$f"(0) = -10$ $f"(2) = 2$

Inc: $(-\infty, 1), (\frac{7}{3}, \infty)$ Dec: $(1, \frac{7}{3})$

CC Up: $(\frac{5}{3}, \infty)$ CC down: $\left(-\infty, \frac{5}{3}\right)$

Max: $(1, 3)$ Min: $(\frac{7}{3}, \frac{49}{27})$ POI: $(\frac{5}{3}, \frac{65}{27})$

The graph of $f(x) = x^3 - 5x^2 + 7x$



Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

4. $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

First derivative

$$g'(x) = -\frac{1}{8}\left[(x+2)^2 2(x-4) + (x-4)^2 2(x+2)\right]$$

$$= -\frac{1}{8}\left[2(x+2)(x-4)(x+2+x-4)\right]$$

$$= -\frac{1}{4}(x+2)(x-4)(2x-2)$$

$$= -\frac{1}{2}(x+2)(x-4)(x-1)$$

Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

4. $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

Second derivative:

$$g''(x) = -\frac{1}{2}[(x+2)(x-4) + (x+2)(x-1) + (x-4)(x-1)]$$

$$= -\frac{1}{2}[x^2 - 2x - 8 + x^2 + x - 2 + x^2 - 5x + 4]$$

$$= -\frac{1}{2}[3x^2 - 6x - 6] = -\frac{3}{2}(x^2 - 2x - 2)$$

Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

4. $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

First: $g'(x) = -\frac{1}{2}(x+2)(x-4)(x-1)$

Second: $g''(x) = -\frac{3}{2}(x^2 - 2x - 2)$

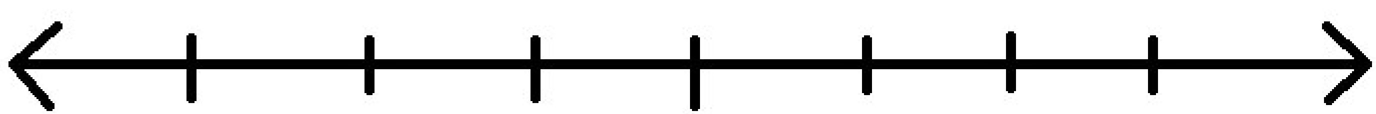
CN: $x = -2, 4, 1$ Possible POI: $x = 1 \pm \sqrt{3}$

Function: $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

First: $g'(x) = -\frac{1}{2}(x+2)(x-4)(x-1)$

Second: $g''(x) = -\frac{3}{2}(x^2 - 2x - 2)$

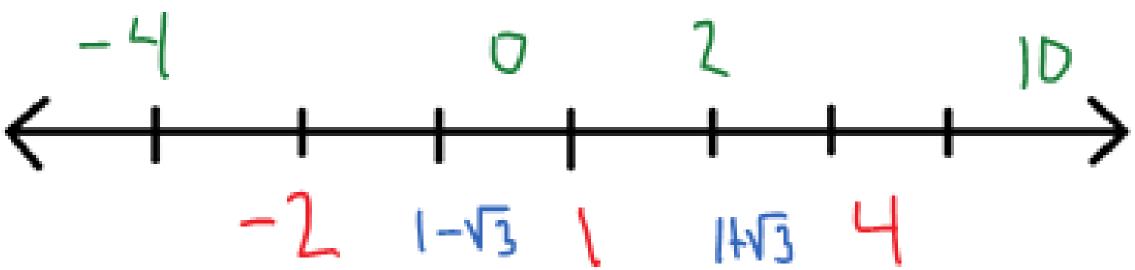
CN: $x = -2, 4, 1$ Possible POI: $x = 1 \pm \sqrt{3}$



Function: $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

First: $g'(x) = -\frac{1}{2}(x+2)(x-4)(x-1)$

Second: $g''(x) = -\frac{3}{2}(x^2-2x-2)$



$g'(-4) = +$ $\quad g'(0) = -$ $\quad g'(2) = +$ $\quad g'(10) =$

$g''(-4) = -$ $\quad g''(0) = +$ $\quad g''(10) = +$

Inc: $(-\infty, -2), (1,4)$ Dec: $(-2,1), (4,\infty)$

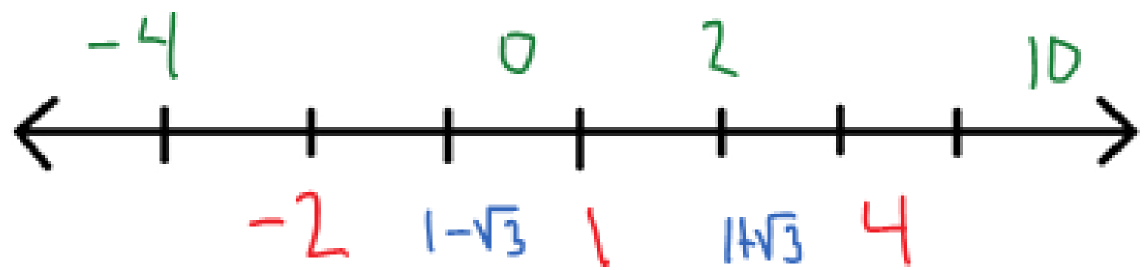
CC Up: $(1-\sqrt{3}, 1+\sqrt{3})$

CC Down: $\left(-\infty, 1-\sqrt{3}\right), (1+\sqrt{3}, \infty)$

Function: $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

First: $g'(x) = -\frac{1}{2}(x+2)(x-4)(x-1)$

Second: $g''(x) = -\frac{3}{2}(x^2 - 2x - 2)$

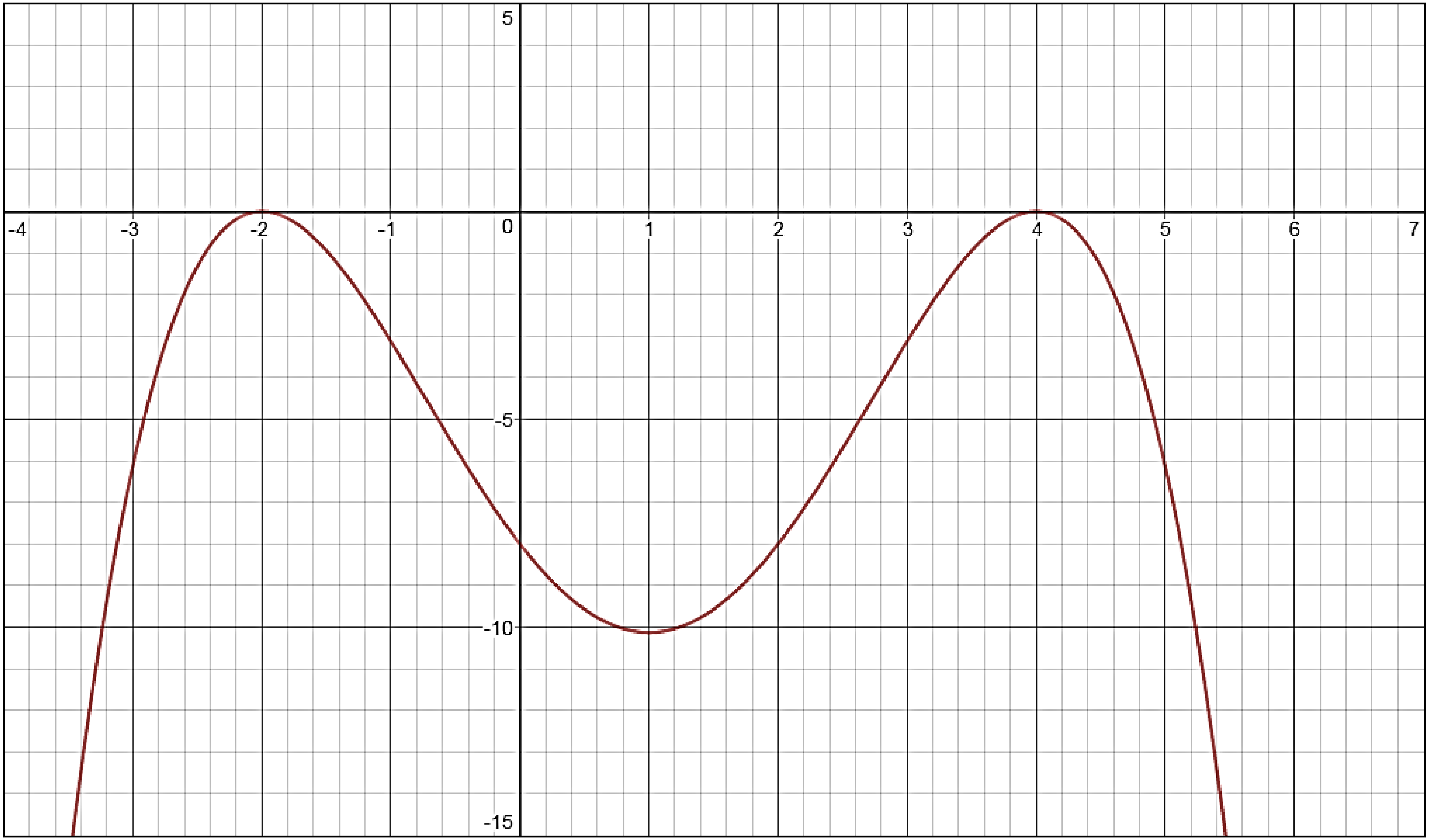


Maximum Points: $(-2,0)$ and $(4,0)$

Minimum Point: $(1, -\frac{81}{8})$

Points of Inflection: $(1-\sqrt{3}, -\frac{9}{2})$ and $(1+\sqrt{3}, -\frac{9}{2})$

The graph of $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$



Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

5. $y = \frac{x}{x-1}$

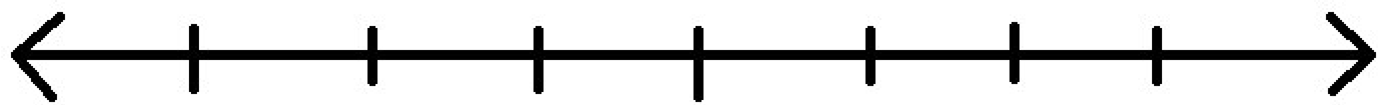
1st derivative: $y' = \frac{(x-1)(1)-x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$

2nd derivative:

$$y'' = \frac{(x-1)^2(0) - (-1)2(x-1)(1)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

The only critical number for either derivative occurs at x = 1. Notice this value is not in the domain of the original function either.

Using $y' = \frac{-1}{(x-1)^2}$ and $y'' = \frac{2}{(x-1)^3}$ and x = 1:



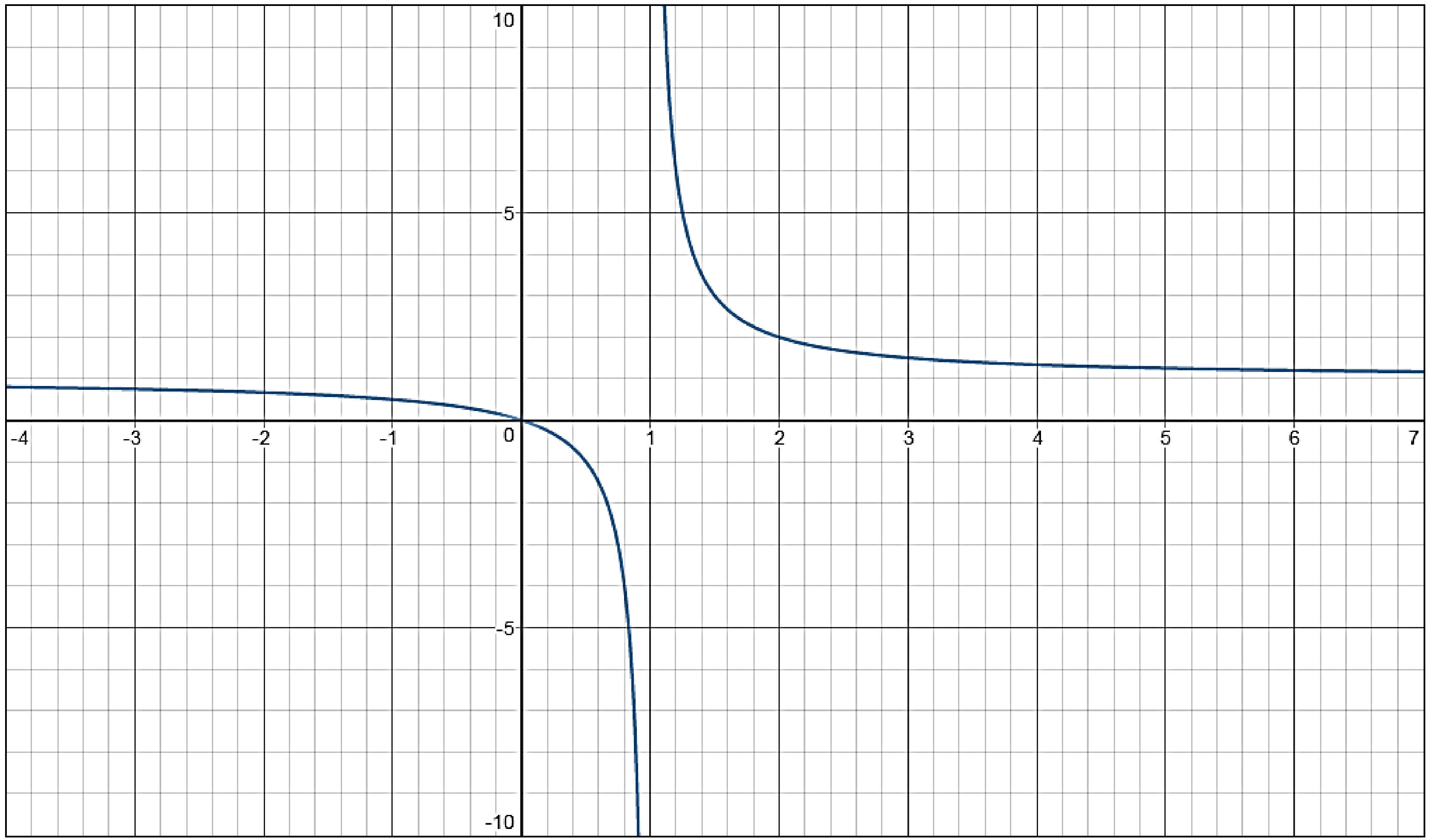
$y'(0) = \frac{neg}{pos}$ $y'(2) = \frac{neg}{pos}$ Dec: $(-\infty, 1), (1, \infty)$

$y''(0) = \frac{pos}{neg}$ $y''(2) = \frac{pos}{pos}$

CC Up: $(1, \infty)$ CC down: $(-\infty, 1)$

No extrema as $x = 1$ is not in the domain of the function.

The graph of $y = \frac{x}{x-1}$



End of Lecture