Limits at Infinity

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Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

Limits at Infinity

Note

- There is a video that accompanies this section on my website and on YouTube. The link is: https://www.youtube.com/watch?v=WKsDmaL8YAM

- In this video I talk specifically about Limits at Infinity and give several examples.

Definition of Limits at Infinity

- Let L be a real number.
  1. The statement $\lim_{x \to \infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
  2. The statement $\lim_{x \to -\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $\mathrm{N} < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Definition of Horizontal Asymptote

The line $y = L$ is a horizontal asymptote of the graph of f if either

$$\lim_{x \to \infty} f(x) = L$$

or

$$\lim_{x \to -\infty} f(x) = L$$

(That is, one limit or the other limit or both.)

Theorem 3.10 (Limits at Infinity)

If *r* is a positive rational number and *c* is any real number, then $\lim_{x\to\infty} \frac{c}{x^r} = 0.$

Furthermore, if x^r is defined when $x < 0$, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0.$

Guidelines for Finding Limits at $\pm$ or Rational Functions

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0. (That is, the horizontal asymptote is the equation $y = 0$.)
2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

Examples: Find the limit

1. $\lim_{x \to (-\infty)} (\frac{5}{x} - \frac{x}{3})$

To find this limit, find the individual limits:

$$\lim_{x \to (-\infty)} \frac{5}{x} = 0$$

$$\lim_{x \to (-\infty)} \frac{x}{3} = DNE$$

Now the arithmetic becomes 0 – $DNE = DNE$. That is, this limit does not exist.

Examples: Find the limit

2. $\lim_{x\to\infty} \frac{x^2+3}{2x^2-1}$

Since x is going to infinity, and is not zero, we can divide each term of the function by the highest power of x in the entire expression:

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{3}{x^2}}{2-\frac{1}{x^2}} = \frac{1}{2}$$

Examples: Find the limit

3. $\lim_{x\to(-\infty)} \frac{x}{\sqrt{x^2+1}}$

Notice that $x = \sqrt{x^2}$ so inside the radical we divide by x^2 and outside the radical we divide by x.

$$\lim_{x\to(-\infty)} \frac{\frac{x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}} = \lim_{x\to(-\infty)} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

Examples: Find the limit

4. $\lim_{x\to\infty} \cos\frac{1}{x}$

As $x \to \infty$, $\frac{1}{x} \to 0$. Therefore,

$$\lim_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1$$

Examples: Find the limit. You try it.

5. $\lim_{x\to\infty} \frac{3x^3+2x-7}{\sqrt{x^4+5x}}$

6. $\lim_{x\to(-\infty)} \frac{\sqrt{x^8-1}}{x^5-1}$

End of Lecture