A Summary of Curve Sketching

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Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

A Summary of Curve Sketching

Note

- There is a video that accompanies this section on my website and on YouTube. The link is: https://www.youtube.com/watch?v=bsb9Opn1420

- In this video I talk specifically about one example of using information from the function and its derivatives to sketch a graph without the use of a graphing calculator

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

Examples: Analyze and sketch

1. $y = -2x^4 + 3x^2$

- From Pre-Calculus, we know this graph has both ends facing the same direction and that direction is down. We could use infinite limits in Calculus to conclude the same thing.
- We can find the x-intercepts:

$$0 = x^2(-2x^2 + 3)$$
$$x = 0, \pm\sqrt{3/2}$$

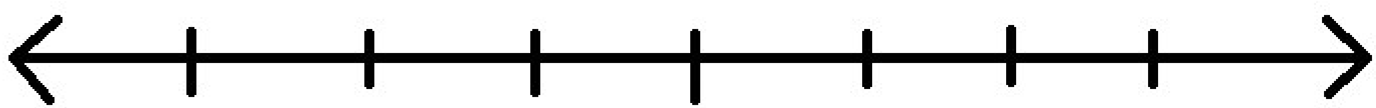
- The y-intercept is at $(0,0)$ and there are no asymptotes.

Examples: Analyze and sketch

1. $y = -2x^4 + 3x^2$

First Derivative: $y' = -8x^3 + 6x$

Critical Numbers: $0 = -2x(4x^2 - 3)$ so $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$.



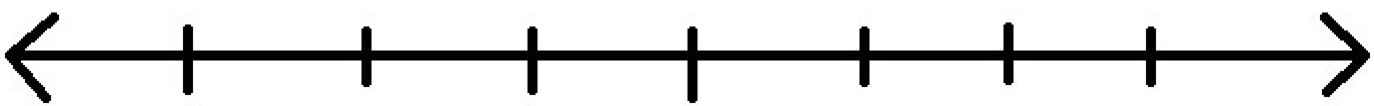
Examples: Analyze and sketch

1. $y = -2x^4 + 3x^2$

First derivative: $y' = -8x^3 + 6x$

Second derivative: $y'' = -24x^2 + 6$

Set to zero: $0 = -6(4x^2 - 1)$ so $x = \pm\frac{1}{2}$

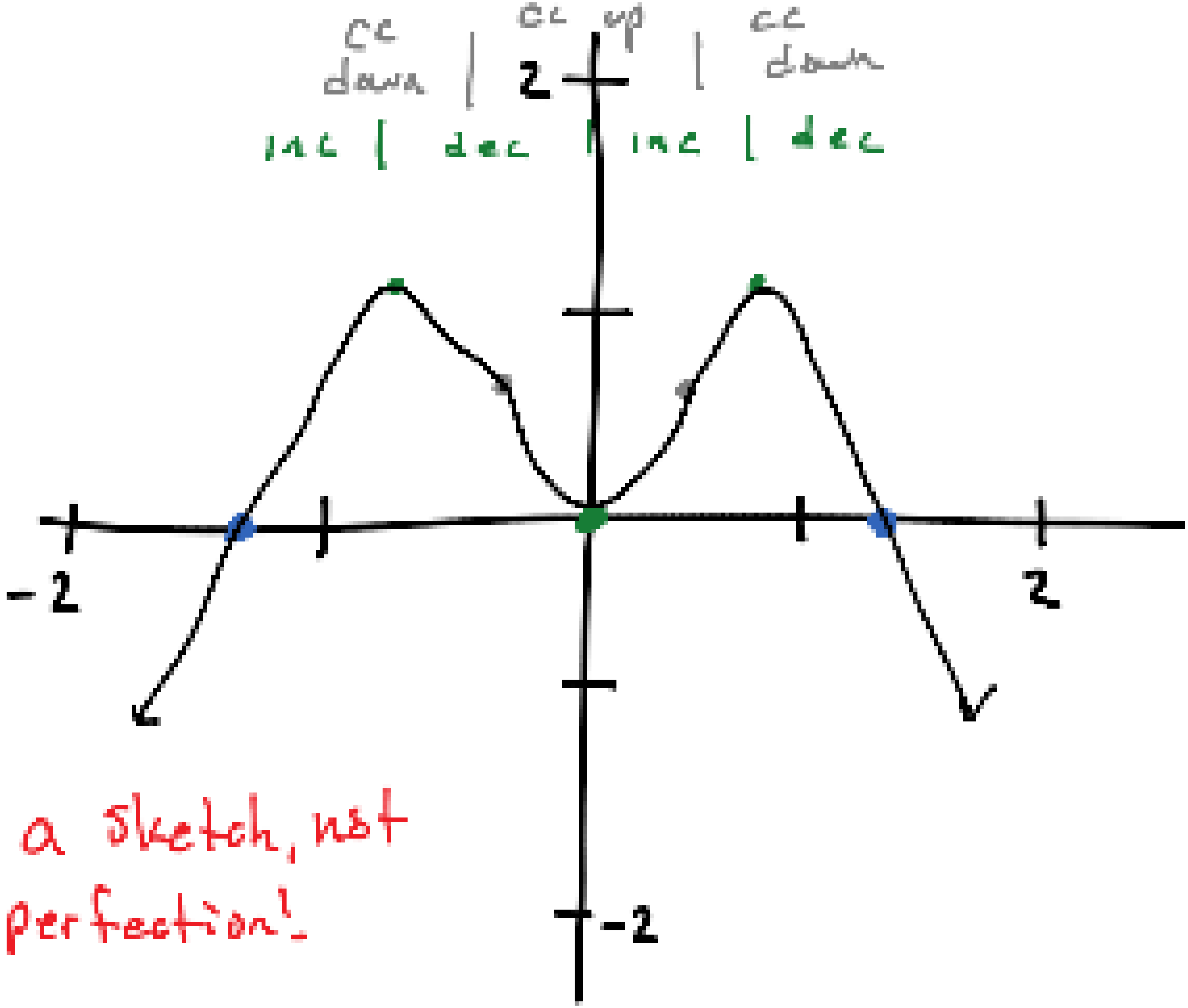


Examples: Analyze and sketch

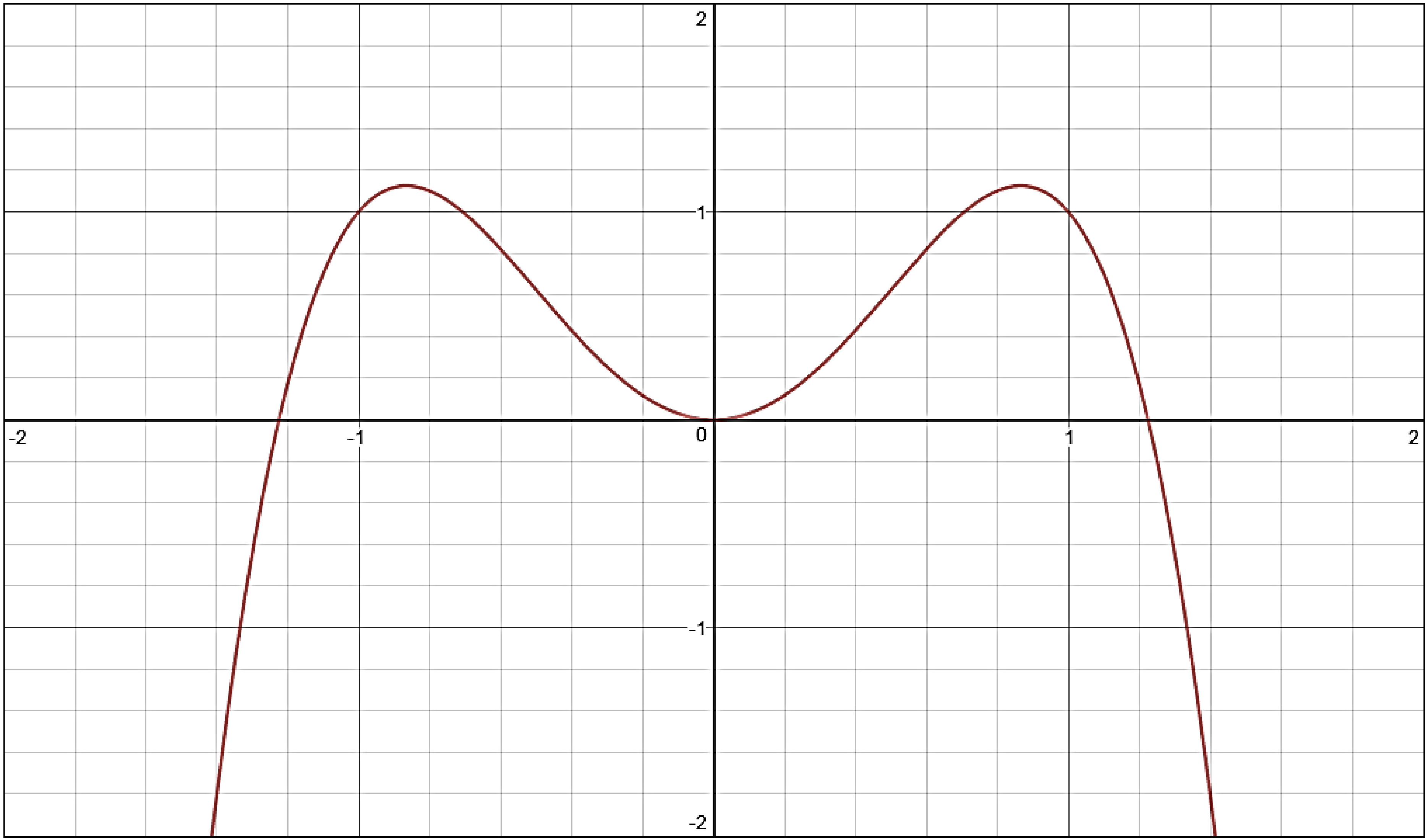
1. $y = -2x^4 + 3x^2$

- We found the locations of some important points using the derivatives so now we find the corresponding y-values using the original function.

- Maximums: $\left(-\frac{\sqrt{3}}{2}, \frac{9}{8}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{9}{8}\right)$
- Minimum: $(0,0)$
- Points of inflection: $\left(-\frac{1}{2}, \frac{5}{8}\right)$ and $\left(\frac{1}{2}, \frac{5}{8}\right)$



a sketch, not perfection!


2
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2

Examples: Analyze and sketch

2. $f(x) = \frac{2x^2-5x+5}{x-2}$

- The numerator is never zero (discriminant is negative) so there is no x-intercepts.
- Vertical asymptote at $x = 2$.
- The y-intercept is at $\left(0, -\frac{5}{2}\right)$.
- No horizontal asymptotes as the limit as x approaches infinity does not exist.
- There is a slant asymptote of $y = 2x - 1$

Examples: Analyze and sketch

2. $f(x) = \frac{2x^2-5x+5}{x-2}$

First derivative:

$$f'(x) = \frac{(x-2)(4x-5)-(2x^2-5x+5)(1)}{(x-2)^2}$$

$$f'(x) = \frac{4x^2-5x-8x+10-2x^2+5x-5}{(x-2)^2}$$

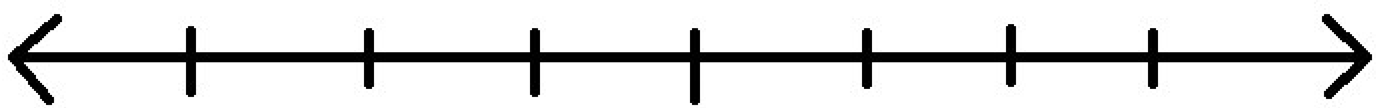
$$f'(x) = \frac{2x^2-8x+5}{(x-2)^2}$$

Examples: Analyze and sketch

2. $f(x) = \frac{2x^2-5x+5}{x-2}$

First derivative: $f'(x) = \frac{2x^2-8x+5}{(x-2)^2}$

Critical numbers: $0 = 2x^2 - 8x + 5$ so $x = \frac{4 \pm \sqrt{6}}{2}$ and $x = 2$



Examples: Analyze and sketch

2. $f(x) = \frac{2x^2-5x+5}{x-2}$

First derivative: $f'(x) = \frac{2x^2-8x+5}{(x-2)^2}$

Second derivative:

$$f''(x) = \frac{(x-2)^2(4x-8) - (2x^2-8x+4)2(x-2)}{(x-2)^4}$$

$$f''(x) = \frac{(x-2)(4x-8) - 2(2x^2-8x+4)}{(x-2)^3}$$

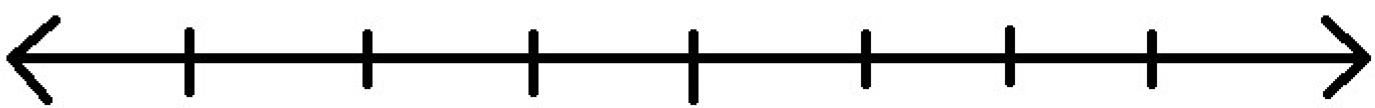
$$f''(x) = \frac{6}{(x-2)^3}$$

Examples: Analyze and sketch

2. $f(x) = \frac{2x^2-5x+5}{x-2}$

First derivative: $f'(x) = \frac{2x^2-8x+5}{(x-2)^2}$

Second derivative: $f''(x) = \frac{6}{(x-2)^3}$



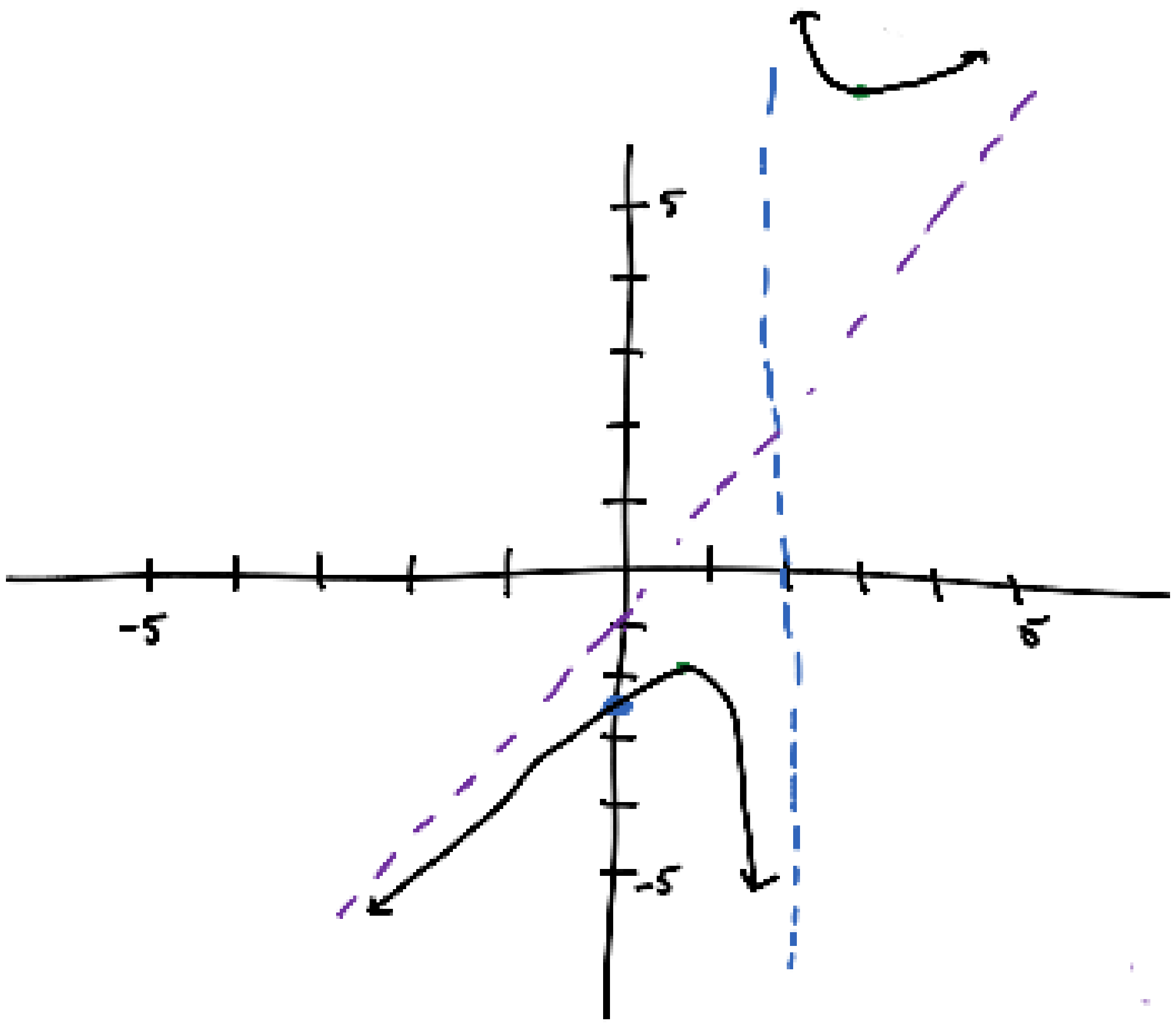
Examples: Analyze and sketch

2. $f(x) = \frac{2x^2-5x+5}{x-2}$

- We found the x-values of key points now we find the y-values from the original function to go with them.

- Maximum: $\left(\frac{4-\sqrt{6}}{2}, 3 - 2\sqrt{6}\right) \approx (0.8, -1.9)$

- Minimum: $\left(\frac{4+\sqrt{6}}{2}, 3 + 2\sqrt{6}\right) \approx (3.2, 7.9)$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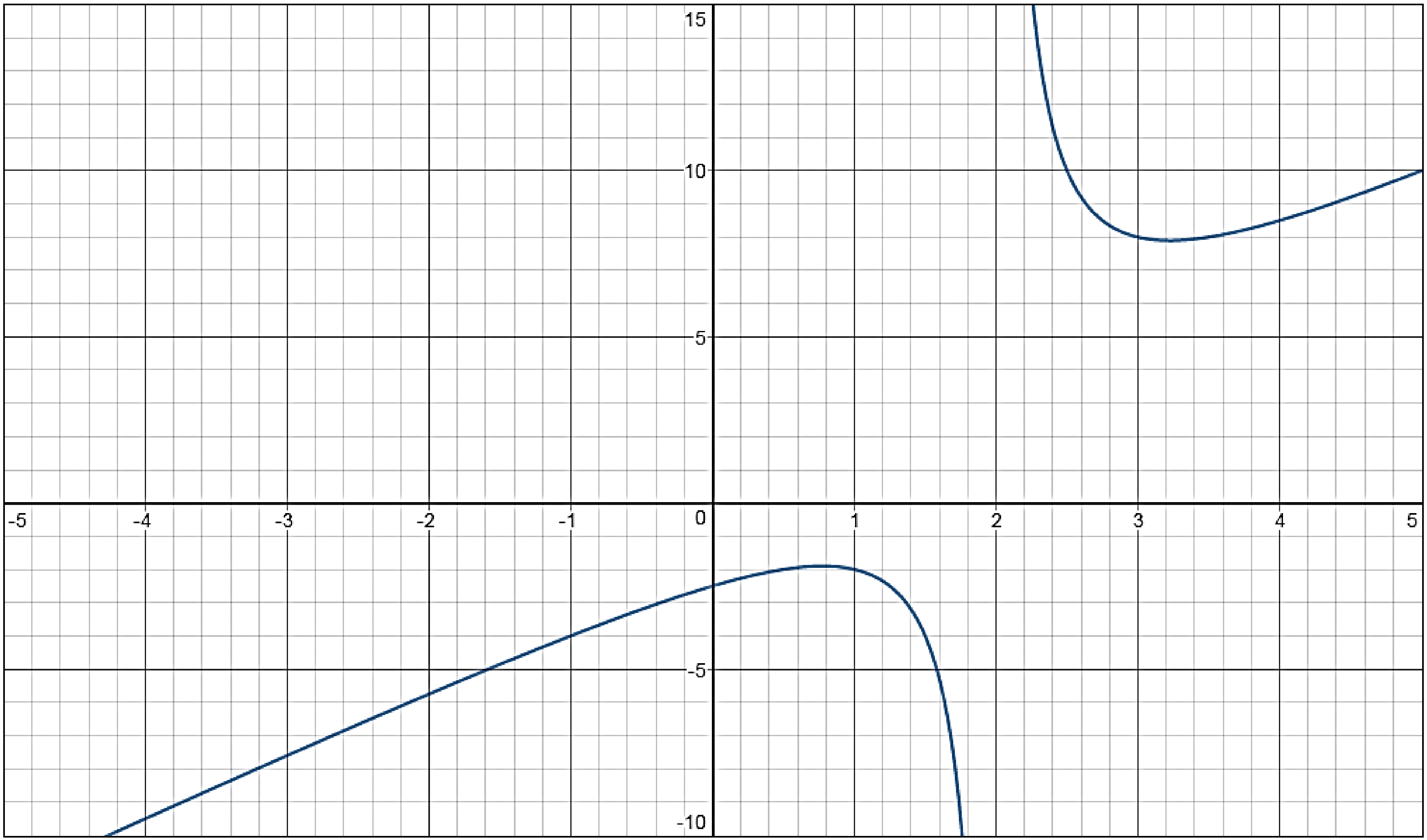
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Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

- From the function we can find the domain:

$$9 - x^2 \geq 0$$

So the domain is $[-3,3]$.

- The x-intercepts are at $x = -3,0,3$ and the y-intercept is a $(0,0)$.

Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

First derivative:

$$y' = x\frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) + \sqrt{9 - x^2}(1)$$

$$y' = \frac{-x^2}{\sqrt{9 - x^2}} + \sqrt{9 - x^2}$$

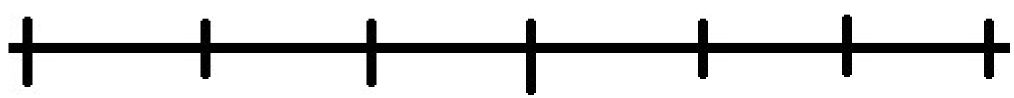
$$y' = \frac{-x^2}{\sqrt{9 - x^2}} + \frac{9 - x^2}{\sqrt{9 - x^2}} = \frac{9 - 2x^2}{\sqrt{9 - x^2}}$$

Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

First derivative: $y' = \frac{9-2x^2}{\sqrt{9-x^2}}$

The derivative is zero when $9 - 2x^2 = 0$ so when $x = \pm\frac{3\sqrt{2}}{2}$. The derivative does not exist at the endpoints of the domain.



Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

First derivative: $y' = \frac{9-2x^2}{\sqrt{9-x^2}}$

Second derivative:

$$y" = \frac{\sqrt{9 - x^2}(-4x) - (9 - x^2)\frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x)}{\sqrt{9 - x^2}^{\,2}}$$

$$y" = \frac{-4x\sqrt{9 - x^2} + \frac{x(9 - x^2)}{\sqrt{9 - x^2}}}{9 - x^2}$$

$$y'' = \frac{-4x\sqrt{9-x^2} + \frac{x(9-x^2)}{\sqrt{9-x^2}}}{9-x^2}$$

$$y'' = \frac{\frac{-4x(9-x^2)}{\sqrt{9-x^2}} + \frac{x(9-x^2)}{\sqrt{9-x^2}}}{9-x^2}$$

$$y'' = \frac{-36x + 4x^3 + 9x - 2x^3}{(9-x^2)^{3/2}}$$

$$y'' = \frac{2x^3 - 27x}{(9-x^2)^{3/2}}$$

Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

First derivative: $y' = \frac{9-2x^2}{\sqrt{9-x^2}}$

Second derivative: $y" = \frac{2x^3-27x}{(9-x^2)^{3/2}}$

The second derivative is zero when $x = 0, \pm\frac{3\sqrt{6}}{2}$

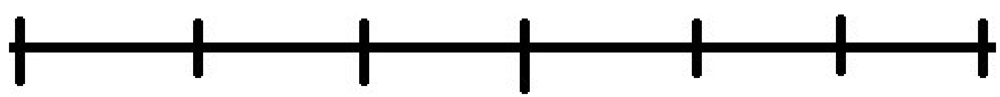
But these last two are outside of the domain so the only one that we focus on is $x = 0$.

Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

First derivative: $y' = \frac{9-2x^2}{\sqrt{9-x^2}}$

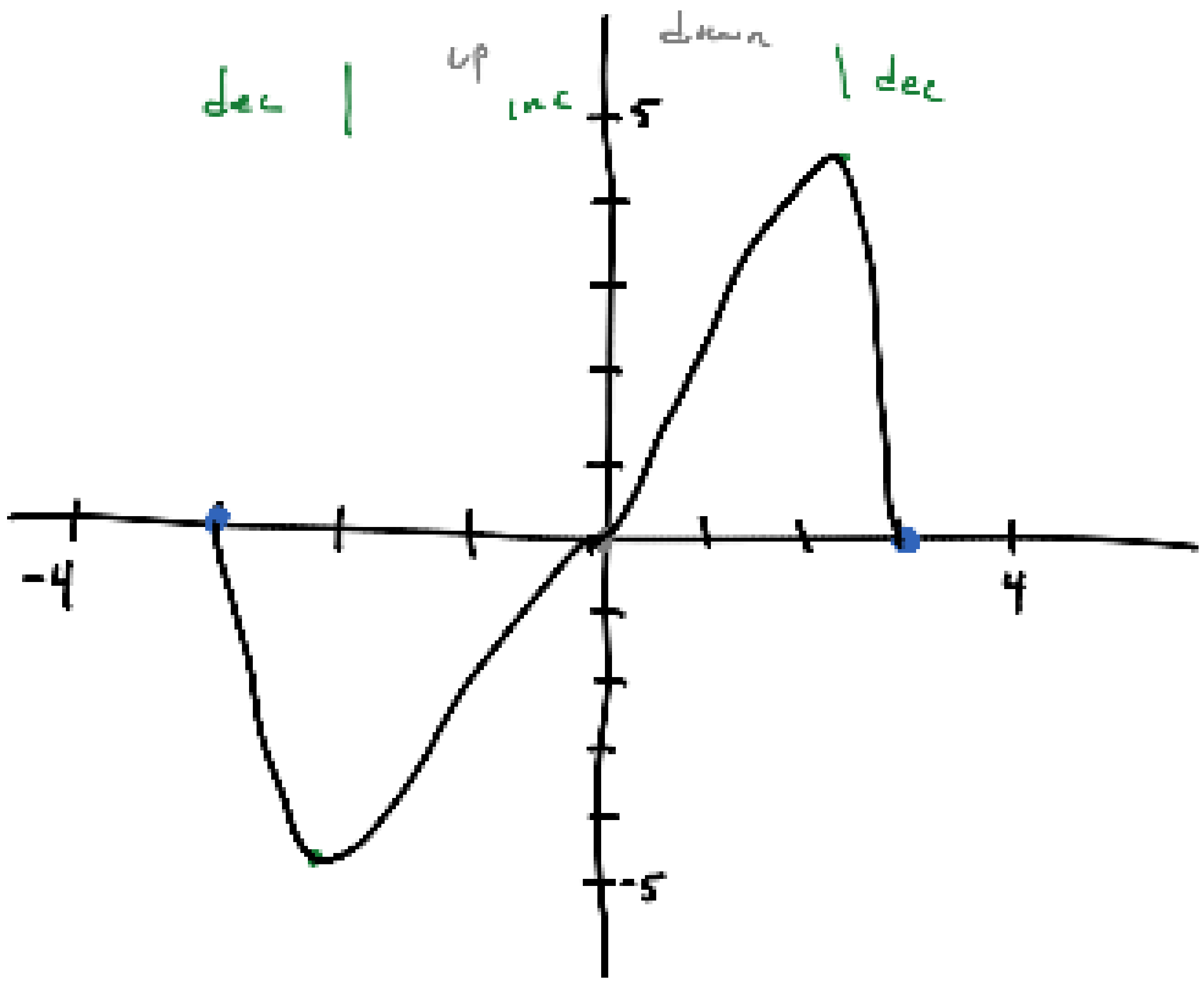
Second derivative: $y" = \frac{2x^3-27x}{(9-x^2)^{3/2}}$



Examples: Analyze and sketch

3. $y = x\sqrt{9 - x^2}$

- We found important locations, now we find the y-values that go with them.
- Maximum: $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$
- Minimum: $\left(-\frac{3\sqrt{2}}{2}, -\frac{9}{2}\right)$
- Point of inflection: $(0,0)$



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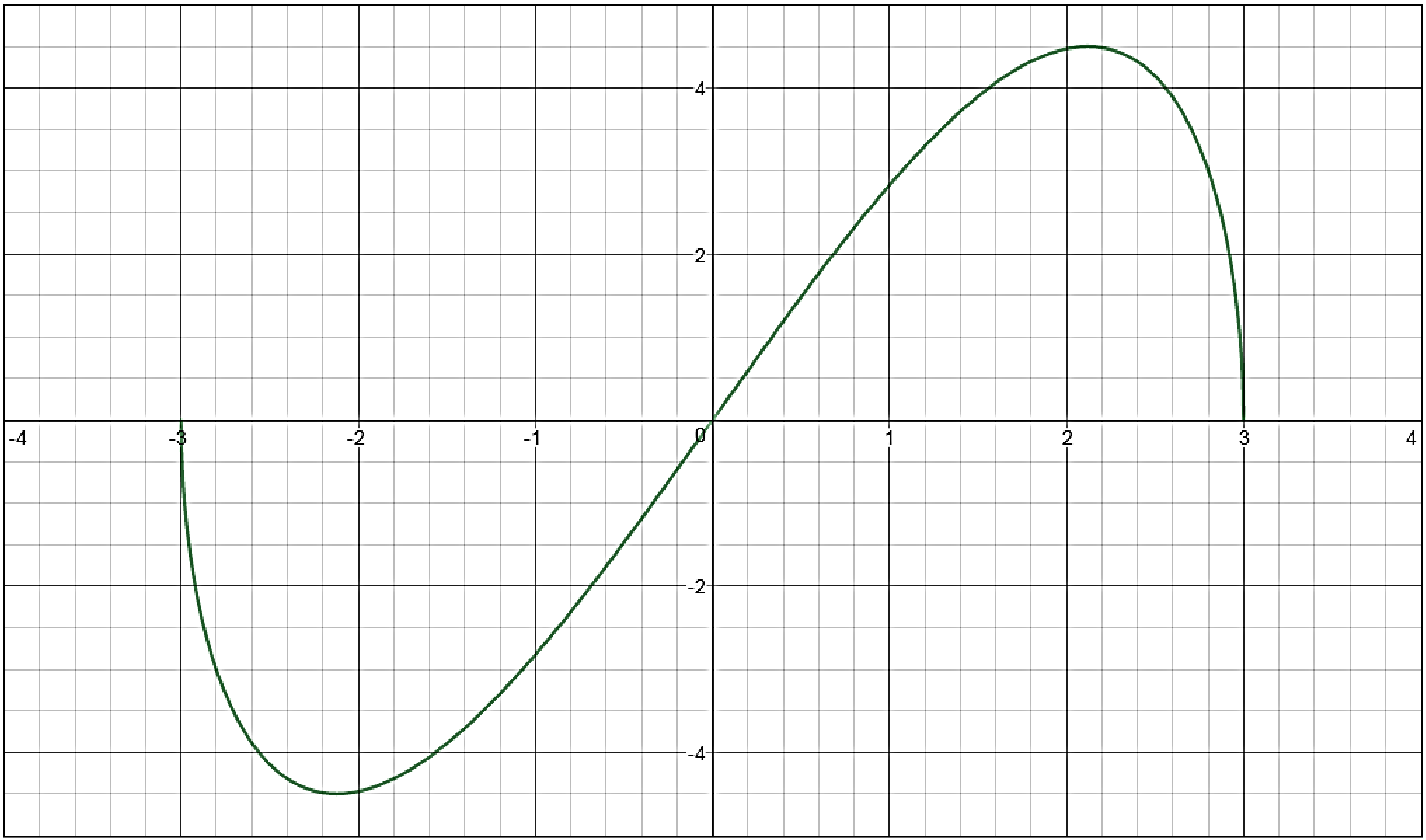
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End of Lecture