Optimization Problems

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Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

Optimization Problems

Guidelines for Solving Applied Minimum and Maximum Problems

1. Identify all given quantities and all quantities to be determined. If possible, or necessary, make a sketch.
2. Write a preliminary equation for the quantity that is to be maximized or minimized.
3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum using the Calculus techniques we have developed.

Find two positive numbers such that the product is 185 and the sum is a minimum.

- Two positive numbers: let them be x and y.
- The product is 185: $xy = 185$.
- Sum is a minimum: $S = x + y$.

The sum equation is our primary equation. We want to use the product equation to help eliminate a variable. If $xy = 185$, then $y = \frac{185}{x}$. We can now focus on S in one variable.

Find two positive numbers such that the product is 185 and the sum is a minimum.

Sum is a minimum: $S = x + y = x + \frac{185}{x}$

The minimum will occur at a critical number so we find the derivative:

$$S = x + \frac{185}{x} = x + 185x^{-1}$$

$$S' = 1 - 185x^{-2} = 1 - \frac{185}{x^2}$$

S' does not exist at $x = 0$, but this is not a positive number so let's solve $S' = 0$.

Find two positive numbers such that the product is 185 and the sum is a minimum.

Sum is a minimum: $S = x + y = x + \frac{185}{x}$

$$0 = 1 - \frac{185}{x^2}$$

$$\frac{185}{x^2} = 1$$

So $x^2 = 185$ and $x = \sqrt{185}$. We only use the positive square root based on the conditions of the problem.

Find two positive numbers such that the product is 185 and the sum is a minimum.

Sum is a minimum: $S = x + y = x + \frac{185}{x}$

First derivative: $S' = 1 - \frac{185}{x^2}$

How do we know $x = \sqrt{185}$ is a minimum?

Second derivative: $S'' = 2(185)x^{-3} = \frac{370}{x^3}$ and $S''(\sqrt{185})$ is positive. By the 2nd derivative test this is a minimum.

If $x = \sqrt{185}$ and $y = \frac{185}{x}$, then $y = \sqrt{185}$ also.

Find the length and width of a rectangle that has a perimeter of 80 meters and a maximum area.

- $P = 80 = 2L + 2W$ or
- $40 = L + W$ or
- $40 - W = L$

- Primary equation is area with

$$A = LW = (40 - W)W = 40W - W^2$$
$$A' = 40 - 2W$$

$A' = 0$ when $0 = 40 - 2W$ so $W = 20$. The second derivative is always negative so this is a maximum.

Find the length and width of a rectangle that has a perimeter of 80 meters and a maximum area.

- $P = 80 = 2L + 2W$ or
- $40 = L + W$ or
- $40 - W = L$
- Primary equation is area with

$$A = 40W - W^2$$
$$A' = 40 - 2W$$

$A' = 0$ so $W = 20$. If $W = 20$, then

$L = 40 – 20 = 20.$

A rectangle of max area will always be a square.

Find the point on the graph of the function $f(x) = (x-1)^2$ that is closest to the point $(-5,3)$.

- "Closest to" tells us to find the minimum distance.
- Using the distance formula and the points $(-5,3)$ and $(x,(x-1)^2)$ we have:

$$d = \sqrt{(-5-x)^2 + (3-(x-1)^2)^2}$$

This does not look fun, the derivative will be easier if we simplify the radicand to a polynomial.

$$d = \sqrt{(-5 - x)^2 + (3 - (x - 1)^2)^2}$$

$$d = \sqrt{x^4 - 4x^3 + x^2 + 18x + 29}$$

Now we can find the derivative to find the minimum:

$$d' = \frac{1}{2}(x^4 - 4x^3 + x^2 + 18x + 29)^{-\frac{1}{2}}(4x^3 - 12x^2 + 2x + 18)$$

$$d' = \frac{4x^3 - 12x^2 + 2x + 18}{2\sqrt{x^4 - 4x^3 + x^2 + 18x + 29}}$$

$$d' = \frac{2x^3 - 6x^2 + x + 9}{\sqrt{x^4 - 4x^3 + x^2 + 18x + 29}}$$

Find the point on the graph of the function $f(x) = (x-1)^2$ that is closest to the point $(-5,3)$.

Minimize: $d = \sqrt{(-5-x)^2 + (3-(x-1)^2)^2}$

With $d' = \dfrac{2x^3-6x^2+x+9}{\sqrt{x^4-4x^3+x^2+18x+29}}$

The derivative will be zero when the numerator is zero…using techniques from Pre-Calculus we find this to be at $x = -1$. (The other two roots are complex).

We could use the 2nd derivative test to see if this is a minimum, or assume we are excellent at algebra and trust it!

Find the point on the graph of the function $f(x) = (x-1)^2$ that is closest to the point $(-5,3)$.

Minimize: $d = \sqrt{(-5-x)^2+(3-(x-1)^2)^2}$

With $d' = \frac{2x^3-6x^2+x+9}{\sqrt{x^4-4x^3+x^2+18x+29}}$

With $x = -1$, $y = (-1-1)^2 = 4$.

The point $(-1,4)$ on $f(x) = (x-1)^2$ is closest to $(-5,3)$

On a given day, the flow rate F (cars per hour) on a congested roadway is $F = \frac{v}{22+0.02v^2}$ where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

- "Maximize flow rate" tells us to find the derivative.

$$F' = \frac{\left(22+0.02v^2\right)(1)-v(0.04v)}{(22+0.02v^2)^2} = \frac{22-0.02v^2}{(22+0.02v^2)^2}$$

$F' = 0$ when $22 - 0.02v^2 = 0$ so $v = \sqrt{1100} \approx$ 33 mph. Test points confirm this is a max.

A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

- Volume of a right circular cylinder and volume of a sphere combine to give 14.
- Minimize surface area.
- We need some formulas!

- $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 14$
- $SA = 4\pi r^2 + 2\pi r h$

- $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 14$
- $SA = 4\pi r^2 + 2\pi r h$
- To minimize surface area, we need to eliminate one of the variables. It seems easiest to eliminate the h. That is, we will solve the volume equation for h:

$$h = \frac{14 - \frac{4}{3}\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3} r$$

- Substitute into the surface area equation and then take the derivative.

$$SA = 4\pi r^2 + 2\pi r h$$

With $h = \frac{14 - \frac{4}{3}\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3} r$

Makes

$$SA = 4\pi r^2 + 2\pi r \left(\frac{14}{\pi r^2} - \frac{4}{3} r \right)$$

$$SA = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2$$

$$SA = \frac{4}{3}\pi r^2 + \frac{28}{r}$$

$$SA = \frac{4}{3}\pi r^2 + \frac{28}{r}$$

Taking the derivative:

$$SA' = \frac{8}{3}\pi r - \frac{28}{r^2}$$

When the radius is 0 the surface area does not exist so let's solve for $SA' = 0$.

$$0 = \frac{8}{3}\pi r - \frac{28}{r^2}$$

$\frac{8}{3}\pi r = \frac{28}{r^2}$ and so $84 = 8\pi r^3$ making $r^3 = \frac{84}{8\pi}$ or $r = \sqrt[3]{\frac{84}{8\pi}} \approx 1.495$ cm.

A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

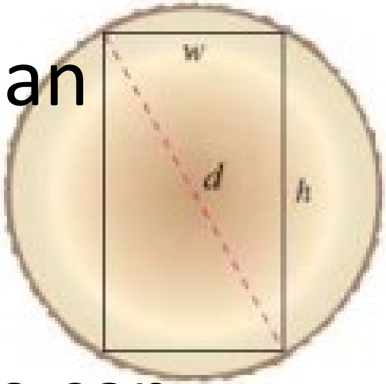
- The radius that minimizes the surface area is $r = \sqrt[3]{\dfrac{84}{8\pi}} \approx 1.495$ cm.

A wooden beam has a rectangular cross section of height h and width w. The strength S of the beam is directly proportional to the width and the square of the height by $S = kwh^2$, where the k is the proportionality constant. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches?

- "Strongest beam" tells us to maximize the strength equation given.
- The value k is a constant, but we have two other variables that we must get down to one.

A wooden beam has a rectangular cross section of height h and width w. What are the dimensions of the strongest beam, $S = kwh^2$, that can be cut from a round log of diameter 20 inches?

It seems as though we have a Pythagorean relationship with $w^2 + h^2 = 20^2$.



Since our equation already has an h^2, we can solve for it and substitute: $h^2 = 400 - w^2$

So now,

$$S = kw(400 - w^2)$$

$$S = kw(400 - w^2)$$

Simplifying we have

$$S = 400kw - kw^3$$

Taking the derivative will give

$$S' = 400k - 3kw^2$$

The critical numbers are when $S' = 0$ so

$$0 = 400k - 3kw^2$$

$$3kw^2 = 400k$$

$$w^2 = \frac{400k}{3k} = \frac{400}{3}$$

Therefore, $w = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3} \approx 11.5$ inches

What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches?

With, $w = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3} \approx 11.5$ inches and $h^2 = 400 - w^2$, it must be that

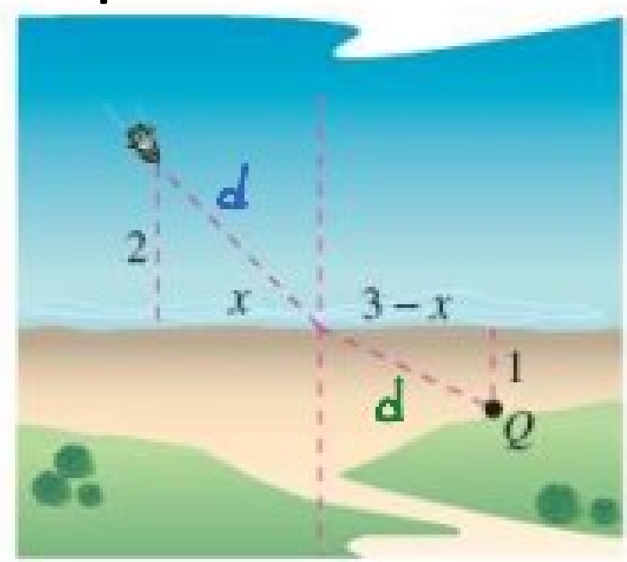
$$h^2 = 400 - \frac{400}{3} = \frac{800}{3}$$

and so $h = \sqrt{\frac{800}{3}} \approx 16.3$ inches.

The second derivative, $S'' = -6kw$ which is negative for all positive widths; this is a max.

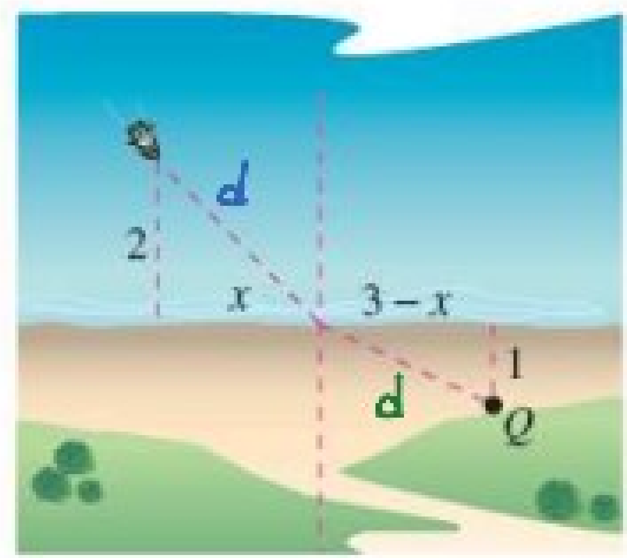
A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland. He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?

- We can break this into two parts:
  - Time on water
  - Time on land
- $D = rt$ so $t = \frac{D}{r}$
- Pythagoras



He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?

- Water: $d = \sqrt{x^2+4}$ so Time = $\frac{\sqrt{x^2+4}}{2}$
- Land: $d = \sqrt{(3-x)^2+1} = \sqrt{x^2-6x+10}$ so Time = $\frac{\sqrt{x^2-6x+10}}{4}$



“least time” means minimize the total time

Total Time = $T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$

The minimum will occur at a critical number:

$$T' = \frac{1}{2}\left(\frac{1}{2}(x^2+4)^{-\frac{1}{2}}(2x)\right) + \frac{1}{4}\left(\frac{1}{2}(x^2-6x+10)^{-\frac{1}{2}}(2x-6)\right)$$

$$T' = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$$

Total Time = $\mathrm{T} = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$

First derivative:

$$T' = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$$

Solving

$$0 = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$$

$$\frac{-x}{2\sqrt{x^2+4}} = \frac{x-3}{4\sqrt{x^2-6x+10}}$$

Square both sides to get…

Total Time = $\mathrm{T} = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$

$$\frac{-x}{2\sqrt{x^2+4}} = \frac{x-3}{4\sqrt{x^2-6x+10}}$$

Square both sides to get

$$\frac{x^2}{4(x^2+4)} = \frac{x^2-6x+9}{16(x^2-6x+10)}$$

Cross multiply to get

$$16x^4 - 96x^3 + 160x^2$$
$$= 4x^4 - 24x^3 + 36x^2 + 16x^2 - 96x + 144$$

Or

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

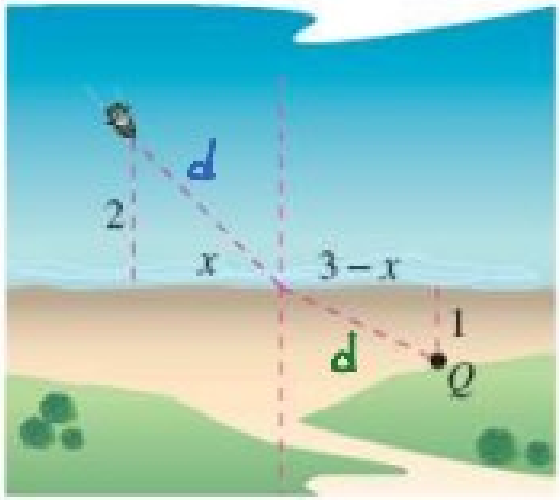
Total Time = $\mathrm{T} = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$

Solving

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

we get $x = 1$ or $x \approx -1.11$.

- Rowing a negative distance would be ridiculous so the man should row toward a point one mile down the coast to minimize his travel time.



End of Lecture