Newton’s Method

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Suggested Review Topics

- Algebra skills reviews suggested:
  - None
- Trigonometric skills reviews suggested:
  - None

Applications of Differentiation

Newton’s Method

Newton's Method for Approximating the Zeros of a Function

- Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.
  1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)
  2. Determine a new approximation
  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
  3. If $|x_n - x_{n+1}|$ is within the desired accuracy, you have your final approximation. Otherwise, return to step 2 and calculate a new approximation.

Facts

- Each successive application of the procedure is called an iteration.

- This is the same process your graphing calculator (which you aren't allowed to use) uses when asked to find a zero.

Examples: Complete two iterations of Newton's Method for the function using the given initial guess.

1. $f(x) = x^3 - 3$ with $x_1 = 1.4$

First, find the derivative: $f'(x) = 3x^2$

For $x_1 = 1.4$ we will have

$$x_2 = 1.4 - \frac{f(1.4)}{f'(1.4)} = 1.4 - \frac{(1.4)^3 - 3}{3(1.4)^2} = 1.4435$$

Now with $x_2 = 1.4435$ we will have

$$x_3 = 1.4435 - \frac{f(1.4435)}{f'(1.4435)}$$

$$= 1.4435 - \frac{1.4435^3 - 3}{3(1.4435)^2} = 1.4423$$

Examples: Complete two iterations of Newton's Method for the function using the given initial guess.

1. $f(x) = x^3 - 3$ with $x_1 = 1.4$

- We have $x_1 = 1.4$, $x_2 = 1.4435$, and $x_3 = 1.4423$.
- We see that both x_2 and x_3 are accurate to two decimal places so our answer would be 1.44.

Examples: Complete two iterations of Newton's Method for the function using the given initial guess.

2. $f(x) = \tan x$ with $x_1 = 0.1$

First, find the derivative: $f'(x) = sec^2 x$

For $x_1 = 0.1$ we will have

$$x_2 = 0.1 - \frac{f(0.1)}{f'(0.1)} = 0.1 - \frac{\tan(0.1)}{sec^2(0.1)} = 0.000\,665\,3$$

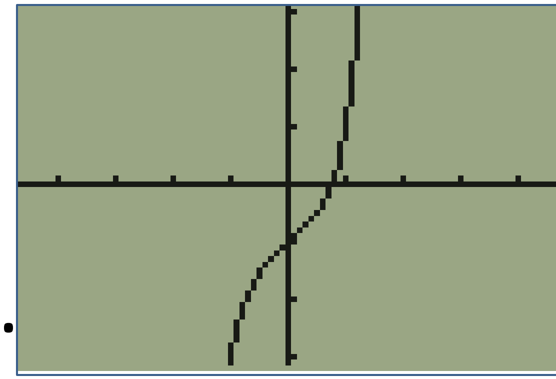
Now with $x_2 = 0.000\,665\,3$ we will have

$$x_3 = 0.000\,665\,3 - \frac{\tan(0.000\,665\,3)}{sec^2(0.000\,665\,3)}$$

$$= 0.000\,000\,000\,196$$

This is a pretty accurate zero.

Examples: Approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001.



1. $f(x) = x^5 + x - 1$

The derivative is $f'(x) = 5x^4 + 1$.

It looks like there is a zero just below x = 1. Let's start there:

$$x_1 = 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{6} = \frac{5}{6} \approx 0.83333$$

1. $f(x) = x^5 + x - 1$

The derivative is $f'(x) = 5x^4 + 1$.

$$x_1 = 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{6} = \frac{5}{6} \approx 0.83333$$

$$x_3 = \frac{5}{6} - \frac{f\left(\frac{5}{6}\right)}{f'\left(\frac{5}{6}\right)} = \frac{10138}{13263} \approx 0.76438$$

$$x_4 = 0.76438 - \frac{f(0.76438)}{f'(0.76438)} \approx 0.75502$$

$$x_5 = 0.75502 - \frac{f(0.75502)}{f'(0.75502)} \approx 0.75487770$$

$$x_6 = 0.75488 - \frac{f(0.75488)}{f'(0.75488)} \approx 0.754877666$$

1. $f(x) = x^5 + x - 1$

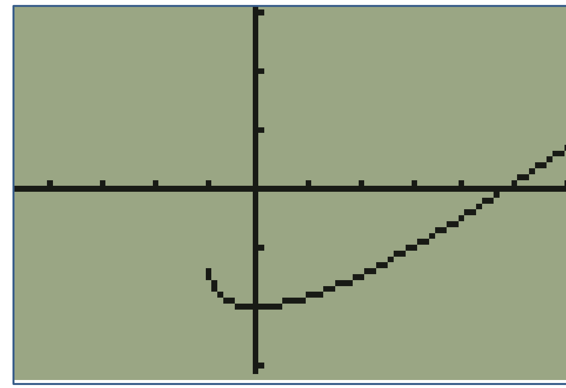
$$x_5 = 0.75502 - \frac{f(0.75502)}{f'(0.75502)} \approx 0.75487770$$

$$x_6 = 0.75488 - \frac{f(0.75488)}{f'(0.75488)} \approx 0.754877666$$

These last two values are within 0.0001 of each other so we will accept x_6 as our solution.

Examples: Approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001.

2. $f(x) = x - 2\sqrt{x+1}$



The derivative is $f'(x) = 1 - \frac{1}{\sqrt{x+1}}$.

It looks like there is a zero near x = 5. Let's start there:

$$x_1 = 1$$

$$x_2 = 5 - \frac{f(5)}{f'(5)} \approx 4.8292856399$$

2. $f(x) = x - 2\sqrt{x+1}$

The derivative is $f'(x) = 1 - \frac{1}{\sqrt{x+1}}$.

$$x_1 = 5$$

$$x_2 = 5 - \frac{f(5)}{f'(5)} \approx 4.8292856399$$

$$x_3 = 4.8292856399 - \frac{f(4.8292856399)}{f'(4.8292856399}$$

$$\approx 4.8284271471$$

$$x_4 = 4.8284271471 - \frac{f(4.8284271471)}{f'(4.8284271471)}$$

$$\approx 4.82842712475$$

2. $f(x) = x - 2\sqrt{x+1}$

$$x_3 \approx 4.8284271471$$
$$x_4 \approx 2842712475$$

These last two approximations are similar to seven decimal places. We will take x_4 as our approximation of the zero of the given function.

End of Lecture

- There is another PowerPoint of Newton’s Method on my website just below this lecture.
- There is also a video quiz that will help you understand Newton’s Method in more detail.