Finding Limits Graphically and Numerically

1. Welcome to finding limits graphically and numerically. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include evaluating functions, graphing functions, working with inequalities, working with absolute values, understanding conjugates and the difference of squares formula.
3. Let's get started with Calculus I Limits and Their Properties: Finding Limits Graphically and Numerically. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 1.2
4. Though this lecture has a focus on limits numerically and graphically, we will also take a look at both the informal and formal definitions of limits. Limits are the building blocks of all of calculus. We will not be using them specifically throughout the course, but as we gain new methods we must keep in mind that the limits gave us those tools.
   Graphing functions seems pretty straightforward for functions that have a domain of all real numbers. We choose a few domain points, find the corresponding range values, then plot and join with a smooth curve. It is when the domain has exclusions, or multiple definitions, that we need to determine what is going on at, or near, these values. In order to do this we use what is called a limit.
5. The informal definition of a limit states that "If f of x becomes arbitrarily close to a single number L as x approaches c from either side, the limit of f of x as x approaches c is L." In this graph we can see that f of x becomes as close to 4 as we want it to be as x gets closer to 2 from either the right side (values greater than 2) or from the left side (values less than 2). We do not care what happens AT x equal to 2. We just want to know what is going on with the y values as the x values get close to 2. This is the difference between a limit and just evaluating a function. Evaluating cares only what is happening at a specific location whereas a limit doesn't care about that one location, just what is happening near it.
6. The notation for a limit is written as L-I-M for limit, then x approaches a value c, of the function f of x, and this is equal to L. Notations are a big part of mathematics. Always know what the notation is saying. Think of the limit as an operator; an operator must have something to operate on. The L I M does not exist without the variable approaching a number c and it does not exist without a function f to operate on. That is, "the limit as x approaches c is 7" has no meaning in mathematics without the function itself.

7. All examples will have a starter list. If you choose to try these on your own before viewing my solutions, this is the slide to use. Feel free to hit pause at any time in order to practice on your own. We will now go over evaluating limits numerically, that is, by using a table of values.
8. For our first example, we will find the limit as x approaches 2 of the function quantity x – 2 divided by the quantity x squared minus 4. Remember that we want to know what is going on near our c value, in this case 2. In order to discover that, we need to look at values that are close to 2 on both sides. That is, values less than (left of on a number line) and values greater than (right of on a number line). Notice the use a parentheses in the table. This is the way you would enter it in your calculator in order to evaluate. The book, and the original statement of the function, rarely uses enough parentheses for a calculator. Your calculator is only going to do what you tell it to, make sure you are telling it the proper order of operations for the given problem. Why did I choose these specific x values? I want to get near 2. Yes, 4 is near 2 and so is 3. But not near enough to give us a good idea of what is happening at 2. I arbitrarily choose these values but could have used 1.95, 1.99634 or whatever I felt like. Notice the question marks at the x value of 2, we cannot plug in this value as 2 is not in the domain of the function. A limit may still exist even though a function value doesn't.
9. Using a calculator to evaluate to four decimal places, we find our values for the function at the chosen values of x below 2.
10. And then we fill in the table with the values of the function for the chosen x values above 2.
11. Looking at both lines of values, I see that as I get closer and closer to 2, it appears that the y values are closing in on 0.25. For this reason, I make a reasoned guess that the limit is 0.25 = ¼. We will use fractions more than decimals in Calculus as fractions are more exact. In this situation, I would accept either answer as would most colleagues and computer homework systems. However, if the answer was 1/3, 0.33 is not equivalent. My general guideline is that if the problem does not ask you to round, use a fraction unless the decimal terminates within 4 places.
12. Example 2 is finding the limit as x approaches negative 5 of the function square root of the quantity 4 – x then subtract three, all in the numerator, divided by the quantity x + 5 in the denominator. We repeat the process of evaluating the function. Notice all the parentheses necessary for this function. We will always surround the numerator with parentheses and the denominator with parentheses, but other places of note include radicands (the thing under the radical) and also exponents which we will see in later lectures. Once again, I do not substitute -5 into the function as -5 is not in the domain. I am looking for a good guess as to what it would be, if it could be.
13. In these tables I have rounded to 4 decimal places, somewhat arbitrarily. Some of my colleagues might prefer 6 or even 8 decimal places. The more decimals showing in the table, the easier it is to determine the limit.
14. As you progress in mathematics, you will want to know several unit fractions for each of conversion. Yes, the limit appears to be -0.166 repeating, but this turns out to be equivalent to the fraction -1/6. The fraction is the better answer for most online homework systems.
15. Next, we find the limit as x approaches 4 of a complex fraction with numerator given by x divided by x + 1 minus 4/5 and denominator x – 4. Once again I have added parentheses in the

table to help you determine how to enter this into your calculator appropriately. I choose values close to 4 but on the left, and I choose values close to 4 on the right.

16. Both rows appear to be going to 0.04 which is fine as a solution, but if you want to write the fraction it is 1/25.
17. Our last example of this type is a trigonometric limit. When using trigonometry in calculus, always use radian mode on your calculator. Degrees are nice for our brains to think in, but radians are real numbers and the truly scientific way of dealing with trigonometry.
    To find the limit as x approaches 0 for sine of 4x all divided by x we choose values close to 0 on either side and evaluate. This one looks pretty nice as both rows look to give us a limit of 4.
18. Next we move to graphing to find limits. If you want to try it on your own first, this is the slide to pause on.
19. In problem one, we want to find the limit as x approaches 2 for the linear function 3x – 4. Our first step is the draw the graph.
20. Looking at the graph, we see the blue arrows getting close to 2 from the left and from the right. As the values on the left get closer and closer to 2, the green squiggly arrow show the function values are getting close to y = 2. As the values on the right get closer and closer to 2, the green squiggly arrow on that side show the function values are getting close to y = 2. It must be that as x approaches 2 on this graph, the limit is also 2.
21. The limit as x approaches negative 2 of the absolute value of x + 2 divided by the quantity x + 2 is a little challenging if you are no familiar with the graph. Notice that the numerator and denominator are the same for values larger than negative 2 (as both numerator and denominator will be positive) and only have opposite value for x values smaller than negative 2. This gives a graph that is -1 to the left of x values of -2 and +1 for x values to the right of -2.
22. Since the graph does not meet up at one specific value for the input of x equals negative 2, this limit does not exist. We will frequently abbreviate "does not exist" as DNE.
23. Example 2 has us finding the limit as x approaches 1 of 5 divided by the quantity x – 1. This is a basic reciprocal graph that has been shifted to the right one unit and vertically stretched by a factor of 5.
24. As x approaches 1, the graph seems to blow up to both positive infinity (from the right) and negative infinity (from the left). This also means that the limit does not exist.
25. The common types of behavior associated with nonexistence of a limit are listed here. First, the function approaches a different number from the right side than it approaches from the left. This was demonstrated in the graph of the absolute value previously. Second, the function increases, or decreases, without bound as x approaches c. That is, the function values go to infinity. Finally, the function oscillates between two fixed values. This frequently occurs with trigonometric functions.
26. Now that you know the basic idea of a limit, it is time for the formal definition. "Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement limit as x approaches c of f of x equals L means that for each epsilon greater than zero there exists a delta greater than zero such that if the absolute value of the quantity x – c is between 0 and delta, then the absolute value of the quantity of the function minus L will be less than epsilon. This is known as the epsilon-delta definition of a limit.

27. Let's take a look at that definition with a graph. You can get within epsilon of the limit of the function be getting within delta of the input value.
28. Let's look at two examples where we first find the limit, either graphically or numerically, and then we will find the delta value necessary for the function to be within 1 one-hundredth of the limit value.
29. The limit as x approaches 2 of 3x + 2 is easily found to be 8 using the graph.
30. To find our delta, we start with the absolute value of the quantity of the function minus L and our tolerance of 0.01. Always keep your goal in mind. Your goal is to find the distance between x and 2, in this case. It will be the distance from x to c in general. With our starting absolute value inequality we can substitute the function rule and the limit L we found. We now have absolute value of the quantity 3x + 2 – 8 is less than 0.01 but we can combine like terms to get the absolute value of quantity 3x – 6 is less than 0.01. Our goal is to get the absolute value of x – 2, so we should factor out the 3 and divide. We now have our goal and it must be that delta is equal to 0.01 divided by 3.
31. With this second example we once again use the graph to find the limit is equal to 29.
32. We keep our goal in mind as well as our starting point prior to substituting known values. The substituting and simplifying are straight-forward. It is the last step that may be somewhat confusing. Notice that x squared minus 25 is just a difference of squares. We can factor a difference of squares. Since this was contained in an absolute value, both factors will be in an absolute value as well.
33. When it comes to solving inequalities, we can multiply or divide both sides by the same thing and obtain an equivalent inequality as long as we don't use zero. We want to divide both sides by the absolute value of the quantity x + 5 in order to isolate its conjugate. We must make sure that this is not zero (or negative). But we know it isn't negative so the order of the inequality will be preserved. Recall that we are looking near 5. There are many values near 5. Four, 4.786, 6 and even 19 are near 5 if you expand your thinking on the word "near." For me, since I know I will be dividing, I choose a value close to 5 but slightly bigger. This will give me a more exact (smaller) answer. After I choose 6, I plug it into the absolute value to get 11; I now am dividing by 11. This makes my delta equal to 0.01/11. There are many other answers depending on your choice of "near."
34. Our last three examples of this section are using the epsilon-delta definition to prove the limit we found is accurate.
35. Graphing, we find that the limit as x approaches negative 3 of 2x + 5 is equal to -1.
36. We use the definition keeping our goal in mind as we make appropriate substitutions and simplifications. In the end we see that we can get our function 2x + 5 as close to the limit -1 as we want (the value of epsilon), as long as our x value is within a delta value equivalent to half of epsilon of negative 3.
37. The graph tells us that the limit as x approaches 4 of the square root function is 2.
38. Using the definition we substitute and then are a little stuck. This is where your algebra knowledge has to help you out. Keep your goal in mind, you want to get to the absolute value of quantity x – 4. How can we get the square root of x minus 2 to turn into x minus 4? Let's multiply both sides of the inequality by the positive quantity absolute value of square root of x

plus 2. This is the conjugate of what we had. Notice this is just going backwards on factoring a difference of squares. After we multiply on the left, we have our goal value. Now we need to handle the right side of the inequality.

39. Near 4 is 4, right? It seems to be the easiest value to substitute in to get an approximation for the absolute value. With this substitution we can get the square root of x as close to 2 as we want (epsilon) as long as delta is four times that value.
40. Our last example is a bit of a let down after all that hard work you've just done. The limit as x approaches 6 of the absolute value of x – 6 is equal to 0. In this case, our starting inequality is the same as our ending inequality and so delta must be equal to epsilon.
41. You can create some of the same graphs I have here anytime you are near a computer by going to desmos dot com and using their free online graphing utility.