Evaluating Limits Analytically

1. Welcome to evaluating limits analytically. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include evaluating functions, rationalizing numerators and/or denominators, evaluating trigonometric functions, and basic quotient and reciprocal identities..
3. Let's get started with Calculus I Limits and Their Properties: Evaluating Limits Analytically. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 1.3.
4. Keep in mind that analytically generally means an algebraic approach. We frequently need some givens to base our work off of and those will be our properties of limits. Proofs of these properties can be found with a basic search, but I will talk through the informal reasoning here. If we let b and c be real numbers and n be a positive integer, the first limit states that as x approaches c, b is just b. Think about the function f(x) = b. What is the output? It is always b. The function doesn't care what x is doing, it always has an output of b and so the limit makes sense. The second limit states that as x goes to c, then x goes to c. It sounds a little odd, but think of the function f(x) = x. Whatever you put in is exactly what you get out. So if the x value goes to c, then the y value does as well. The third limit is based off of the second and does require a more formal proof than "look at what happens" but you can easily convince yourself of its probable validity by looking at several examples.
5. Our second look at properties of limits has the added conditions that the limit as x approaches c of f of x is equal to L and the limit as x approaches c of g of x is equal to K. The first properties states that we can multiply a scalar, a constant value, by the function and this in turn multiplies the scalar by the limiting value L. The sum or difference rule states that if you want to find the limit of a sum (or difference) you can just take the sum of the limits (or difference). The third rule is the product rule; this states that if you want to take the limit of a product of functions, the result is the product of the limits. Notice that this is exactly what we would want to happen in all these situations as they are the easiest possible outcomes.
6. We had addition, subtraction and multiplication properties, so it should be no surprise that we also have a quotient, or division, property. If you take the limit of a quotient of functions, the result is the quotient of the limits. However, and this is a big however, this is only true if the denominator is not zero. If you take a limit and get a denominator of zero then something else must be done in order to evaluate it. The power property is an extension of the earlier power on a variable property but it now extends to powers on entire functions. That is, if you take the limit of a function raised to a power, you get the limit raised to the same power. These 8 limits lead us to some great results that make our job of evaluating limits easier so that we do not

have to rely on the time consuming method of tables or even of knowing what the graph looks like.

7. If we put together the idea of sum and differences along with scalar (constant) multiples, limits of constants, and power rules, we find that we can now take limits of any polynomial. Add in the quotient rule and we can also find limits of all rational functions, as rational functions are defined to be one polynomial divided by another (nonzero) polynomial. Pay attention to the small words here: the limit of a polynomial can be found by evaluating the polynomial. That's it! If you want to take the limit of a polynomial, just evaluate it. Similarly, if the limit of a rational function can be found by evaluating. As long as you do not get a zero denominator. These facts just told us that the majority of the limits we need to find can be found by evaluating.
8. Limits also pass through roots (which are just rational powers) and compositions. For root functions, we again just evaluate in order to find the limit. For the compositions, if f and g are functions such that the limit as x approaches c of g of x is equal to L, and the limit as x approaches c of f of x is equal to f(L), then the limit of f composed with g of x as x approaches c is f evaluated at the limit as x approaches c of g of x which is equal to f(L).
9. Moving past our algebraic functions and into trigonometry, we find that once again limits are really nice. They work out how we would want them to if we got to decide. That is, for any trigonometric function, as long as c is in the domain the limit as x approaches c is equal to the trig function evaluated at c.
10. It may seem like there is a lot to remember and it is time to break out the notecards to keep track of it all. But do not overwhelm yourself. In general, as long as c is in the domain of the function, you can evaluate the function to find the limit. If c is not in the domain of the function we need some other strategies in order to find the limit. Know the rule so that you can deal with the exceptions.
11. Theorem 1.7 (catchy name isn't it?) allows us to deal with some of those exceptions. Let c be a real number and let f(x) and g(x) be the same except at x = c in some interval containing c. That is, the functions are essentially the same close to c. If the limit of g(x) as x approaches c exists, then the limit of f(x) also exists and they are equal. This is nice. The limits of similar functions are the same. But this is actually more powerful that you might first realize. This theorem allows us to simplify a function into something similar, except at c, and then find a limit.
12. Let's talk strategies. First, and really the most helpful, learn to recognize which limits can be evaluated by direct substitution. Second, if the limit of f of x as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than at x equals c. Common techniques involve factoring and canceling or rationalizing. For the function g that you found that is similar, except at x = c, find the limit as x approaches c for g and this will be the same as the limit as x approaches c of f. As always, you can use a table of values or a graph to reinforce your conclusion.
13. If you would like to pause here and try these limits on your own, I encourage you to do so.
14. Always evaluate a polynomial to find the limit. It is easy to make this more complicated; many times you need to remind yourself to not overthink the math involved. For the first problem we substitute 1 everywhere we see an x to get a limit of zero. On the second problem, remember that limits can go inside the radical and therefore we can evaluate the limit of a radical function

by evaluating the function, assuming the value c is in the domain of the radical function. In this case we find the limit is the cube root of 8. Always simplify when possible, whether a fraction or a radical, we will report this answer as 2.

15. Rational functions can be daunting, but if the value of c is in the domain, we simply evaluate the function to find the limit. In this case, the limit as x approaches negative 3 is negative 2. Many trigonometric functions can also be evaluated to find the limit, assuming c is in the domain of the function. For problem 4 we see that tangent is defined at pi and therefore the limit as x approaches pi of tangent of x is equal to zero. Think about what this means; as x gets close to pi, the y values of the tangent function get close to zero.
16. Number 5 is another trigonometric function and the cosine has a domain of all real numbers which allows us the evaluate cosine at 5pi/3 in order to find the limit of ½. Number six is a bit more involved. We would love to evaluate and have an answer. When we evaluate we end up with 0 over 0. Anytime, and every time, you evaluate a limit and get an answer of 0/0 you should do something. In this case I have decided to factor and cancel. This is the process of finding a similar function, except at x = -1, in order to use theorem 1.7 and find the limit. For a problem such as this we might report the limit to be negative 5 and the similar function to be g of x equals 2x minus 3, as shown in blue.
17. Evaluating straight away leads us to 0/0 so we must do something. Many times you will need to know special formulas in order to factor either the numerator or denominator of a rational function. For problem seven we use the difference of cubes formula to factor the numerator. Once cancelled we have a similar function, x squared plus 2x plus 4, in which we evaluate to find the limit of 12. Problem 8 we once again start by trying to evaluate. Fortunately when we evaluate we get a "good" answer of 3/3 or 1. A "good" answer is one that is not the indeterminate form of 0/0.
18. In this last example if we try to evaluate we get 0/0. This means we must do something to find a similar function g. Factoring looks hard, and completely not worth the effort, so we try the opposite and multiply. Really that is all that rationalizing is; we are multiplying a numerator (or denominator) by a specific value so as to clear the radical from the numerator (or denominator in other problems). But we can't multiply by just anything; we must multiply by the exact right thing for the radical to cancel out in the numerator. This will nearly always be the conjugate of the radical expression you already have. In this problem you can see the conjugate written in green. Notice that the conjugate is the perfect thing to multiply by to get the difference of squares formula and no longer have a radical in the numerator. If you need to take a minute and multiply this out completely, pause the video and do it to convince yourself that what I have done is correct. Notice after simplification the original denominator is cancelled out. We have found a similar function g of x equal to 1 divided by the quantity of square root of x plus 1 added to 2.
19. To find the original limit, we can now find the limit of this similar function which is ¼. Several of these examples have led us to this study tip: always try to evaluate the limit by evaluating the given function. If you get a number, great. If you get 0/0 there is always something that can be done to find another function g in order to use theorem 1.7.

20. The Squeeze (or Sandwich) Theorem states that if you have three functions with h of x less than or equal to f of x which in turn is less than equal to g of x for all x in an open interval containing c, except possibly at c itself (basically saying that f is squeezed in between h and g in this interval), and if both h and g have limits of L as x approaches c, then f must have the same limit at c.
21. You can find justifications for these special limits quite quickly and easily online, including on the Larson website, listed at the end of this video. I encourage you to write these down and then memorize as soon as you can. The limit as x approaches zero of sine of x divided by x is 1 and the limit as x approaches zero of the quantity 1 minus cosine x all divided by x is equal to zero.
22. Let's use these special limits to evaluate some trigonometric limits. Pause here if you would like to try on your own first.
23. For problem 1, recall that limits can pass through constants. The three is now multiplied by the special limit and so the limit is 3 multiplied by 1 which is 3. Problem 2 encourages a simplification of trig functions first. Once you write cosine of theta multiplied by tangent of theta you see that can be simplified to sine of theta. We know that has theta approaches zero, sine of theta divided by theta will be 1. Finally, we can rewrite the secant of phi as 1 divided by cosine of phi. Since cosine is not zero at pi, we can evaluate this and not have to use any special limits or formulas. The limit as phi approaches pi of phi times secant of phi is equal to negative pi.
24. This is the end of the lecture on evaluating limits analytically. If you would like to look over Bruce Edward's proof of the special limits, you can find them on the Larson website as listed.