Continuity and One-Sided Limits

1. Welcome to continuity and one-sided limits. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include evaluating functions, rationalizing numerators and/or denominators, evaluating trigonometric functions, and basic quotient and reciprocal identities.
3. Let's get started with Calculus I Limits and Their Properties: Continuity and One-Sided Limits. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 1.4.
4. It is always a good idea to start with a definition. What is continuity? In previous courses you may have heard a continuous function explained as one that you can draw the graph of without lifting your writing implement. That is, it has no holes, breaks, jumps or anything else of that type. This is still true but now we define it in mathematical terms using our limits. The definition states that a function f is continuous at a value x = c if the following three conditions are met. First, the function must be defined at x = c. Second, the limit as x approaches c of f of x must exist. The importance of the third condition is too frequently overlooked. This condition states that not only does the function value exist and not only does the limit exist, but they MUST be equal to each other. For a function to be continuous on an open interval, it must be continuous at each point in the interval.
5. Sometimes it is beneficial to see what happens when conditions are NOT met. On this slide we see from the graph on the left that a hole appears if the function is not defined at a specific x value. This is not a continuous function. The middle graph shows that the function is defined, as the solid dot indicates, but the limit does not exist. Finally, the graph on the right shows that the function value exists, the limit exists, but they are not equal. All three of these graphs are graphs of functions that are not continuous.
6. Sometimes a function will have different behavior to the left and right of the value c under question. When this happens we will look at one sided limits. That is, a limit just on the right of the given value. Or a limit just on the left of the given value. In order to do so we will need a notation. Our notation comes from the standard number line that everyone is used to with negatives on the left and positives on the right.
7. The notation we see here shows a superscript of a positive sign to indicate we should only look to the right of the value c. What this means is that we only look at values above c numerically, or to the right of c on a standard number line or on the x-axis of a rectangular coordinate system. The superscript of a negative sign indicates we should only look at what is happening as the x values get closer and closer to c from the left.

8. We’ve been talking about limits for a couple of sections already. But only now are we able to state when a limit will exist. In fact, a function f can only be continuous at the real value c if the limit from the right is equal to the limit from the left. Another way to think of this is that the limit exists overall if and only if the limits from each side exist and are equal to each other.
9. For this random graph, the blue portion shows the x values to the left of c and the green portion shows the values to the right of c. As x approaches c from the left (I encourage you to use your finger to actually start from the left side of the graph and follow f of x until the x value gets to c.) we see that the graph gets close to the y value of 2. However, if we follow the green portion from the extreme right of the graph and move it toward c, we see that the graph gets close to the y value of negative three. Since the limit from the right is not equal to the limit from the left, the overall limit does not exist for f of x at x = c in this graph.
10. We started this lecture defining continuity at a point and on an open interval. Now that we have one-sided limits we can define continuity on a closed interval as well. A function f is continuous on the closed interval from a to b it if is continuous on the open interval from a to b and the limit from the right as x approaches a is equal to f of a and the limit from the left as x approaches b is equal to f of b.
11. If you would like to try these problems before I go over them, pause the video and try them now.
12. Every time we want to find a limit, we start by trying our previous tricks. On number one we first try to evaluate. This results in zero divided by zero which is undefined. When we get zero over zero we know something can be done. In this case we can factor a negative one from the numerator and use the difference of squares formula on the denominator. Canceling the common factor allows us to try and evaluate once again. The fact that this limit was approaching 2 from the right had no bearing on the answer. But that is not always the case. In problem two we are approaching 10 from the left. The numerator and denominator have the same factor, but the numerator will never be negative. This means that they will always have the same value with possibly different signs. When we approach 10 from the left we are using values smaller than 10 so when we subtract ten from a smaller number the denominator will be negative. This will give us positive divided by negative of the same number resulting in the negative one answer. Keep in mind that if we tried to substitute ten at the start we would end up with zero divided by zero again. This is why zero over zero is called an indeterminate form; in problem one it simplified to be -1/4 and in problem two it simplified to -1.
13. Whether finding an algebraic or trigonometric limit we always want to start by evaluating to see what happens. In the case of cotangent, we know this as the quotient identity cosine divided by sine. But when we evaluate at pi, we get -1 divided by 0. Since we have a non-zero numerator and a zero denominator, this limit does not exist. In this situation there is nothing we can do to “simplify away” the zero in the denominator. Looking at the graph of cotangent we see an asymptote at x equals pi with the limit from the left going to negative infinity and the limit from the right going toward positive infinity. As the one-sided limits are not equal, no limit exists.
14. Evaluating the limit of any function requires you to recognize the type of function it is in the first place. Recall the greatest integer function, with the graph given, has output of the largest integer that is less than or equal to the input. Pre-Calculus notes on my website cover this

function in section 1.6. As we approach 4 from the right (the superscript + sign), the values of x are a little larger than 4. The greatest integer of values "a little larger" than 4 is 4. Upon evaluation we find the limit to be 13.

15. Piece-wise defined functions are only tricky if you let them be. For the most part, you can evaluate the limit of a piece-wise function by evaluating the function itself. The exceptions occur when you have a non-continuous piece or at cut values. Cut values are where the domain cuts from one piece to the next. In this example, that is at the x value of 3. Notice that the limit has x approaching 3 from the left. Which piece of this function should we use? We will only consider the top piece of the function as it has a domain for x values less than or equal to 3. Finally, to actually evaluate the limit we substitute the x value of 3 in the top rule and end up with a limit of 5/2.
16. Now that we know what continuity is and how to determine if a function is continuous, let's look at a few properties. For any real number *b* with function f and g continuous at x = c, then any scalar multiple of one of those functions is also continuous. The sum and/or difference of continuous functions is also continuous. The product of continuous functions is continuous. Finally, the quotient of continuous functions is continuous as long as you have a non-zero denominator. All of these properties taken together mean that most elementary functions are continuous on their domains. The key points to consider are values not in the domain.
17. Pause here if you would like to try these examples on your own before I go through them.
18. Our goal is to find a constant, called *a*, in order to make this piece-wise defined function continuous everywhere. By initial inspection, we know each piece is continuous on its domain. The only value we really must consider is the cut-value; if the function is made to be continuous at the cut value then it will be continuous on the entire real line. In order to be continuous at the cut value, the limit from the left must be equal to the limit from the right and these must be equal to the function value itself.
19. First, we evaluate the limit as x approaches 1 from the left using the bottom piece (x values less than 1) to get a limit of a – 3. Second, we evaluate the limit as x approaches 1 from the right using the top piece (x values greater than or equal to 1) to get a limit of 2.
20. Notice that the limit from the right is the same as the function value since it's domain had the "or equal to" in the restriction. For this function to be continuous it must be that a – 3 = 2 and so *a* must be 5.
21. Again we look at the function in pieces. The top piece is continuous everywhere except at x = 0, but it is not defined at x = 0 so it is continuous on its domain. The bottom piece is a linear function and so is continuous on its domain. The only place this function may not be continuous is the cut value. We evaluate the limit from the left of 0 to get 9. Notice this used a special limit that we learned in an earlier section. We evaluate the limit from the right of 0 to get *a*. This means that *a = 9*.
22. In this problem we are looking for two constants, both *a* and *b*. Our strategy will be to take this one cut value at a time and see what happens.
23. At the x value of -3, we use the top two pieces; the limit from the left is 4 and the limit from the right is -3a –b. If this function is going to be continuous at x = -3 we must have -3a – b = 4. This doesn't allow us to solve for *a* or *b* so let's try the same thing at the cut value of x = 2 using the

bottom two pieces. The limit from the right is -4 and the limit from the left is 2a – b. To be continuous it must be that 2a – b = -4. Again, no information about a or b specifically. However, we have two equations and two unknowns so we can solve this as a system of equations.

24. Solving this system of equations is an algebra problem, not a calculus problem, so review as needed. In this particular case we find that a = -8/5 and b = 4/5 in order to make the function h of x continuous on the entire real line.
25. Once again, pause here if you would like to attempt to work on these problems prior to hearing my explanation. It is always a good idea to try some examples on your own rather than having me work them all for you. We learn from mistakes, but rarely from successes. We wish to find the x values, if any at which the given functions are not continuous. Remember, most elementary functions are continuous on their domains. This means we need to check values that are NOT in the domains of the given functions.
26. To determine if a discontinuity is removable, we need to know that removable discontinuities are the ones in which we can factor and cancel or otherwise deal with the indeterminate form of 0 divided by 0. Our first example has a domain of all real numbers except x = 2. When we check the one-sided limits at x = 2, we find that the limit from the left approaches negative infinity and the limit from the right approaches positive infinity. Therefore this function is not continuous at x = 2 and it is not removable (in fact, it is an asymptote).
27. Polynomials are defined on all real numbers and are also continuous on all real numbers.
28. Again, we check for values where the function is undefined. For rational functions, this is where the denominator is zero. However, x squared is always 0 or greater and then when you add one you find that this denominator is 1 or greater and therefore is never zero. Since the domain consists of all real numbers, the function is continuous everywhere.
29. When setting this denominator to zero and solving we find discontinuities (values where the function is undefined) at plus and minus 1. The limits as we approach either plus or minus one have the form of a non-zero value (specifically 1 or -1) divided by zero. This is not a removable discontinuity.
30. We frequently just look at the denominator of a rational expression, but the numerator is important too. If we simplify (factor) the rational function that is number 5 we find that x minus 6 is divided by the product of x plus 6 and x minus 6. There are discontinuities at both x equals 6 and x equals negative 6. However, the discontinuity at x = 6 is removable as you can cancel those factors to "simplify" the function. Another way of thinking about this is when you evaluate the limit as x approaches 6 you get zero divided by zero so something can be done at it is removable. However as x approaches -6 you get a non-zero number divided by zero as the limit and it is therefore not removable.
31. Piece-wise defined functions look more complicated that other functions, but taken a piece at a time they are many times easier to deal with. We know that each piece is a polynomial and every polynomial is continuous on its domain. Therefore we only have to check the cut value of x at 1.
32. Checking the limits from both the left and right we find the limit exists, the function value exists, and they are all 1.
33. Therefore this function is continuous everywhere.

34. Our first fact on this page tells us that continuity behaves nicely through composition. This is due to the fact that limits behave “nicely” through composition of functions.
    The Intermediate Value Theorem (which I abbreviate as IVT) is an important theorem. Most theorems are important, but the ones with actual names are always going to be the most important. The intermediate value theorem states that if you start with a continuous function on a closed interval where the heights of the endpoints are not equal and *k* is some number between the heights of the endpoints, then there must be at least one number *c* in the closed interval such that f evaluated at c is equal to k.
    One quick way to think of this is driving. As I start in my car at 0 mph I get up to the speed limit of 75 mph and set the cruise control. My driving is a continuous function, the closed interval is a set amount of time that I’m driving and the endpoint heights are the speeds of 0 and 75 mph. The IVT states that there had to have been a time (the c value) at which I was going exactly 55 mph, the k value.
35. Let’s look at two examples of verifying that the intermediate value theorem applies and then finding the c value. The first will be a polynomial and the second a rational function.
36. Our first function is f of x equals x squared minus 6x plus 8 and we are focusing on the closed interval from 0 to 3. This interval is the x values that we are considering. If the IVT applies, we will want to find a c value, between 0 and 3, such that f evaluated at c is equal to zero. Our first step is to make sure the IVT applies. (a) Since the function is a polynomial we know it is continuous on its domain. (b) Next, we check the values of the endpoints. F evaluated at 0 is 8 and f evaluated at 3 is -1 so the values of the endpoints are not equal. (c) Finally, the k value of zero is between 8 and -1 so we know the IVT applies.
37. Finding the c value such that f of c is equal to zero amounts to evaluating the function at c and setting it equal to zero. We can factor c squared minus 6c plus 8 as the product of c minus 4 and c minus 2. Keep in mind this is still and equation and so still equal to zero. We know use our algebra skills of the zero factor property which states that if two things multiply to be zero then one or the other (or both) is zero. This tells us that c = 4 or c = 2. Both answers are tempting; beware of the original problem. We have to keep our results in the closed interval from 0 to 3 and 4 it outside that. The only answer we would report is c = 2.
38. For the rational function, we once again start by verifying that it fits the conditions of the intermediate value theorem. This is a rational function with a discontinuity at x = 1 (where the denominator is zero). However, the only discontinuity is NOT in the given interval. Therefore this function is continuous on the given closed interval. If we evaluate the endpoints we get outputs of 35/6 (a little less than 6) and 20/3 (a little more than 6). The k value of 6 given in f of c equals 6 is between the two endpoint heights and so this function with its closed interval and k fit all the necessary conditions of the intermediate value theorem.
39. Our next goal is to find c. We evaluate the function at c and set it equal to 6. Multiply both sides of the equation by the denominator of c minus 1 to arrive at a polynomial equation of c squared plus c equals 6c minus 6.
40. Setting this equation equal to zero, then factoring, and finally solving for c yields two answers once again. Make sure you check the original interval as c = 2 is NOT a solution but c = 3 is.

41. One last example. Suppose at 8 AM on Saturday a man (a crazy man it sounds like) begins running up the side of a mountain to his weekend campsite. On Sunday morning at 8 AM he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes (because he's super smart like that) that he passed the same place at exactly the same time on Saturday. Prove that he is correct.
42. Let's remember that math problems are frequently wacky to make for good examples and put common sense aside. I have chosen to graph the two days with Saturday in purple starting at base and running up to camp and Sunday in green starting at camp and running down to base. Notice that since he does not have teleportation powers, his run is a continuous line from start to finish. Even though the times are different, his trail from Saturday did cross his trail from Sunday at some time in the first ten minutes. This is where, and when, he was in the same place at the same time.
43. This marks the end of continuity and one-sided limits. Thank you for your time.