Infinite Limits

1. Welcome to infinite limits. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include evaluating functions, graphing functions, working with inequalities, working with absolute values, evaluating trigonometric functions, and graphing trigonometric functions.
3. Let's get started with Calculus I Limits and Their Properties: Infinite Limits. This lecture corresponds to Larson's Calculus, 10th edition, section 1.5.
4. Previously we have seen limits that diverge; that is, limits that go off toward either positive or negative infinity. This usually happens near an asymptote. We said that those limits did not exist. Now we can clarify a bit with infinite limits. Let f be a function that is defined at every real number in some open interval containing c, except possibly at c itself. The statement that the limit as x approaches c of f of x equals infinity means that for any large real number M that is positive, there exists a delta greater than zero such that we can make the function values larger than M as long as the x values are within delta of c. (Notice that the M is replacing the epsilon in our previous definition.) Similarly, we can define limits of negative infinity when there exists a negative number N such that the function values are less than N whenever x is within delta of c.
5. It is important to have one-sided limit that equal negative or positive infinity as well, and those definitions take into account the x values close enough (within delta) only on the appropriate side. A couple of notes to keep in mind. First, having a limit equal to infinity does not mean the limit exists. In fact, it means the limit is unbounded and therefore does not exist. Secondly, for those of you using WebAssign, if the limit is infinity, enter the symbol for infinity as your answer and not DNE. Read the instructions carefully to make sure you are answering as they want however.
6. In Pre-Calculus we found vertical asymptotes by finding the domain of the function and comparing the numerator and denominator. (See the notes to section 2.6 in Math 1508 on my website for a review.) Now with limits, we can formalize the process and definition. If a function approaches infinity, either positive or negative, as x approaches c from the right OR the left, then the line x = c is a vertical asymptote of the graph of the function. Please note, an asymptote is ALWAYS an equation of a line and never just a value. We can also express a theorem that corresponds to the notion we had in Pre-Calculus: Let f and g be continuous functions on an open interval containing c. If f of c is not zero but g of c is zero, and there exists and open interval containing c such that g is not zero at any of the values near c in this interval, then the graph of the function given by h of x equals f of x divided by g of x has a vertical asymptote at x = c. A less formal statement says that if the denominator is zero and the

numerator is not, then you have a vertical asymptote at the value that makes the denominator zero.

7. Let's look at some examples. In using the theorem I will analyze the numerator and denominator separately. I know that the numerator of negative 4x is only zero at x = 0. The denominator is never zero for real values of x as x squared is 0 or greater and when you add 4 the denominator will always be 4 or greater. Since the denominator is never zero, there is no vertical asymptote.
8. Once again we analyze the numerator and denominator separately. The numerator is zero when s = 3/2. The denominator is zero at plus and minus 5. The zeros of the numerator do not correspond to the zeros of the denominator so your vertical asymptotes are s = 5 and s = -5. Notice that these are equations with the appropriate input variable and not just the values plus and minus 5.
9. In order to analyze the numerator, we need to factor it completely using the sum of cubes formula. This is one that you should have committed to memory (or a notecard near you). After we factor we find that both the numerator and denominator have the same zero at x = -1. The theorem says that the numerator cannot be zero for a vertical asymptote to exist therefore this rational function does not have a vertical asymptote. This particular situation results in a removable discontinuity.
10. I include this example as many of my online students have difficulty in expressing the solution, but not in finding it. The hardest part is the details! First we factor both the numerator and denominator completely in order to find the zeros. The numerator has zeros at t equals 0 and 2. The denominator has zeros at t equals plus or minus 2. Remember, the vertical asymptotes only occur where the denominator IS zero but the numerator is not. In this case that means that t = -2 is the only vertical asymptote and t = 2 is a location of a removable discontinuity. This is not the difficult part; everyone seems to get here. But remember, vertical asymptotes are always equations of lines and equations have a left side an equal sign and a right side. Your answer is that the vertical asymptote is t = -2. The entire equation, not just the value -2.
11. No set of examples would be complete without a trigonometric example so let's find the vertical asymptotes, if any for the function given by f of x equals secant of pi*x. You may have the graph of the secant memorized, if so that is fantastic. Your mathematical life will be much easier! But for those of you that have only focused on the "big 3" trig functions of sine, cosine, and tangent, it may be easier to use a reciprocal identity to write your function as 1 over cosine of pi x. In this form it looks like a rational function and so we can see the numerator is never zero. We know the graph of cosine is zero at pi over 2 and 3 pi over 2 and so on, which can be reported as pi over 2 plus n pi, for any arbitrary integer n. But we don't have the function cosine of x, we have cosine of pi x so we know that pi x must be equal to pi over 2 plus n pi for integers n. Solving for x we find that the vertical asymptotes of this function will occur at x = ½ + n for all integers n.
12. Next, we will evaluate some one-sided limits. If you have a calculator, or patience, tables are a great way of evaluating limits if you are unsure and do not have access to a graph. In this table I only looked at values to the right of 1 and I got closer working right to left. Notice how the y values get big and negative as you read them moving toward 1 from the right. This means that the limit as x approaches 1 from the right of the function quantity 2 + x divided by quantity 1 – x

is negative infinity. If you do have access to a graph, as I have included, you can see that the closer we get to 1 approaching from the right only, the graph takes off toward negative infinity.

13. In the previous example I did not evaluate first, as I saw a discontinuity at x = 1. But it should be one of our first instincts all the time. In this example we try to evaluate the rational expression in order to evaluate the limit at we find that the limit is ½. We could use a table or graph instead, but let it be easy when you can. Always try to evaluate first.
14. If we try to evaluate -2 over cosine of x at pi over 2 we get -2/0. But this is undefined. It tells us we have an infinite limit, but it does not tell us if it is positive or negative infinity. This is where we need to look closer at the limit and see that we are approaching pi over 2 from the right. A table is a great idea, but how do you choose radians "close enough to" pi over 2 on the right? A graph is a great way of examining limits. Knowing your 6 basic trig graphs will be extremely helpful throughout calculus.
15. The graph of -2 times secant of x is the graph of secant, reflected vertically and vertically stretched by a factor of 2. As we approach pi over 2 from the right, looking at the graph on the slide, we see that the blue line of the function takes off toward positive infinity. Therefore this limit is equal to positive infinity. Notice that answer would have been different had we approached pi over 2 from the left.
16. Infinity is not a number, it is a symbol for "larger than any number" but it still has properties. These properties will come to us in the form of limits. If you have one function with a limit as x approaches c that is infinite and another function with a limit as x approaches c that is a finite value L, then the sum or difference of the limits will be infinity (watch for signs). The product of a value L and infinity is positive infinity is L is positive and negative infinity if L is negative. If you have a constant in the numerator with the denominator going to infinity, the quotient is zero. All these same properties will hold if we look at one-sided limits as well.
17. This is the end of the lecture. Again, if you would like a free online grapher please check our desmos dot com.