The Derivative and Tangent Line Problem

1. Welcome to the derivative and tangent line problem. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include working with difference quotients, evaluating functions, finding limits and writing linear equations given a point and a slope
3. Let's get started with Calculus I Differentiation: The Derivative and Tangent Line Problem. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 2.1.
4. What is a tangent line? You may be used to talking about tangent lines on circles in which the tangent line touches the circle in only one point. This is a traditional tangent line. But in Calculus we will see curves where our tangent line may touch the curve many times. However locally, near one specific point (as shown with the red dot and line in the graph on the right) the tangent line touches the graph only once. The most important part of this tangent line is that if you zoom in enough on the graph of any curve (as shown in the bottom graph), the tangent line will approximate the curve at that one point. That is, we can use a basic line to tell us many things about more complicated curves.
5. These tangent lines, and more importantly their slopes, come to us via a pre-calculus notion of the difference quotient. Recall that the difference quotient is a generalized slope formula given by the difference f of x plus h minus f of x all divided by h. This difference quotient, and another version we will see later, form the basis for what is essentially half of calculus: the derivative.
6. If f is defined on an open interval containing c, and if the limit as the distance between x values goes to zero of the change in y divided by the change in x, which can be represented as the limit of h approaching zero of the difference quotient at the point c, exists as a real number m, then the line passing through the point with input c and output f of c and slope m is the tangent line to the graph of f at the point c comma f of c.
   The slope of the tangent line is also called the slope of the graph. This limit, the limit of the difference quotient, gives us a slope of a tangent line. Keep that in mind throughout this section and those to come.
7. Our first example will be what I will call a simplistic example. It is one that once we reach the answer we think, but of course that's what it is! These examples serve a good purpose however. Any principle or technique should work just as well for the "easy" problems as they do for the more challenging. For the function given by f of x equals 3/2 x plus 1 we want to find the slope of the tangent line at the point (-2,-2). In order to do this we must find the limit of the difference quotient. I find it easier to find the pieces of the difference quotient and then take the limit

appropriately. First we find f evaluated at -2 + h as our c value is given as -2. Evaluate the function making sure to distribute and combine like terms to get the expression -2 plus 3/2 h. We could evaluate the function at -2 to arrive at the output of -2, but notice we are given the point so I just did extra work for no reason. Keep an eye on that, if they give you the point with both input and output, you can do less algebra along the way.

8. Now that we have our pieces, we can evaluate the limit as h goes to zero. Simplify your numerator first, then the fraction, and then evaluate the limit. The slope of the tangent to the graph of our function at the point (-2,-2) is 3/2. Here we can see why this is an "easy" problem. We know that the given function was a linear function and so it is its own tangent. Therefore the slope of any linear function is also going to be the slope of the tangent line.
9. Let's step it up with a quadratic function. Let g of x equal 6 minus x squared at the point 1 comma 5. We are looking for the slope of the tangent line that touches this parabola at that point. Our c value is given as 1 so we evaluate g at 1 + h and at 1. Notice that we must square appropriately and make sure to distribute the negative sign before combining like terms for g of 1 plus h. In the statement of the problem we were given a y value so we know that g(1) = 5. We substitute these pieces into the limit of the difference quotient. First we simplify the numerator, then we factor in order to cancel with the denominator, and finally we evaluate the limit to get the slope of the tangent line is -2.
10. Sometimes it is nice to see that we are correct. Though we haven't talked about equations of tangent lines yet, you can see that this is the graph of g of x and the slope of the line tangent at the point (1, 5) is indeed -2. I love it when the graph verifies the algebra.
11. In our final example of this type, r of t is equal to t squared plus three at the point -2 comma 7. Using a c value of -2 as given, we evaluate the function at -2 plus h and again at -2. Keep in mind that these are pesky algebra details. Make sure you work them out as we go, but my focus here is the calculus involved. Once you have the pieces you can form your limit of the difference quotient, simplify your numerator, factor and cancel from your denominator, and finally evaluate the limit to get the slope of the tangent line to be -4.
12. We've been calling this limit the slope of the tangent line. That is true, but we have a better calculus word for it. The value of the limit of the difference quotient, the slope of the tangent line, is the derivative of the function at the given point. We could use a general point x or a specific point c. If this limit exists, the derivative for a general point x is also a function.
    As calculus was developed independently by at least two different guys, we have a variety of notations for derivatives. All should be recognized as interchangeable.
13. Now we want to find the derivative function, no input c is given. In step one we evaluate f of x plus h to get 3x plus 3h plus 2. In step 2 we just rewrite f of x to have it near. My step three used to be in the limit of the difference quotient, but I like to do it prior to writing the limit in order to make my limit a little easier to manipulate. Let's find the difference part of the difference quotient making sure to distribute the negative sign. This leaves us with a numerator of 3h. When we substitute into the derivative formula, we can find that the limit is equal to 3. Yes, the derivative is 3. But remember the derivative is a formula for the slope of the tangent line. Since this was a linear equation to start, we knew the slope should be 3 for every tangent line.

14. Again we take the limit definition in steps. We find g of x plus h by substituting x plus h everywhere we see an x. Be sure to use the FOIL process so you do not forget any terms and then distribute the subtraction. We could rewrite the g of x function here, but since it is given in the statement of the problem, that would just be extra work. Simplifying the numerator of the limit definition we find that g of x plus h minus g of x yields only the terms that have an h in them. That is, the difference in the numerator is negative 2x minus h squared.
15. With this numerator we can use the limit definition and find the derivative. First we factor an h out of the numerator in order to cancel from the denominator and then take the limit as h approaches zero of what remains. As h goes to zero, negative 2x minus h becomes negative two x. That is, g prime of x, the derivative function, is equal to negative two x.
16. Our third example looks like a simple enough function. When we evaluate f of x plus h we get 4 divided by the square root of the quantity x plus h. That seems easy. Finally! Something easy. But be weary. Anytime the beginning is "easy" it will get more difficult at some point. When we subtract the function we end up with a difference of two rational expressions. We need a common denominator in order to subtract. Here comes the algebra.
17. Our common denominator is the square root of x multiplied by the square root of the quantity x plus h. Multiplying each rational expression appropriately, in red, we have an equivalent difference with a common denominator which can be combined into a single quotient. Are we done? Is that all we needed to do? Let's check.
18. The derivative is the limit of this fraction divided by h. But when we evaluate we get 0/0. This is not bad, but certainly not good. We aren't finished as anytime, and every time, we get 0/0 there is something else we can do. In this case we should probably rationalize the numerator.
19. Multiplying the numerator and denominator by the conjugate of the numerator in order to use the principle of difference of squares. Simplify the numerator but I choose to leave the denominator in its extended glory. We are able to simplify all the way to negative sixteen h in the numerator with that ugly denominator. Does this help? Can we find the derivative yet?
20. Let's try the limit again. Remember we are taking the limit as h approaches zero of the difference quotient. We have been working with the difference doing all this algebra, don't forget that the denominator of it all is h. But dividing by h is the same as multiplying by 1 over h and that will cancel the h from the numerator fraction. Once we actually let h be zero, we no longer write the limit notation (just like after we add 3 + 4 we no longer write the plus sign). Simplify and we find that the derivative of our original function 4 over square root of x is negative 2 over the product x times the square root of x.
21. Our next set of examples asks us to find the equation of the tangent line to the graph of f at the given point. Think about the words used in the instructions: equation of the tangent line. Since it is an equation, we must have something on the left an equal sign and something on the right. Tangent leads us to derivative since the slope of the tangent line is the derivative at the point. Finally, the word line tells us to find slope-intercept form $y = mx + b$. We know that m is the value of the derivative so we need to find b. Another, possibly better, formula for use with calculus is the point-slope form $y – f(c)=f'(c)(x-c)$ and then simplify to slope-intercept form. Note: On WebAssign the y = may already be present outside the answer box. Other times it is not. Pay close attention to this.

22. To find the equation of a line, I frequently start with finding the slope. In calculus that means finding the derivative and in this section that means finding the derivative using the limit definition. I like to take it in pieces so let's find f of x plus h first by replacing x with x plus h in the original function. Be sure to distribute appropriately to get the expanded form. Next, since we already know f of x, we do the difference part of the difference quotient. When we subtract f of x from f of x plus h we find that the x squared, the 3x and the 4 are canceled out. This will be a clue that you did the first step correctly. Once you find the difference everything that remains should have an h in it and nothing should remain that doesn't. Note, this is a rule for polynomials only. Now that everything has an h in it for the difference, the quotient dividing by h will cancel one power of h from each term that remains. Having found the difference quotient it is now easier to evaluate the limit.
23. The derivative of f at x is equal to the limit as h approaches zero of 2x plus h plus 3. But as h goes to zero, 2x and 3 do nothing and the h becomes zero. This leaves us with a derivative function f prime of x equal to 2x plus 3. Keep in mind we were given the point -2 comma 2 so we evaluate the derivative at -2 in order to find the slope of the tangent line as -1. Knowing the slope and the point, we can now find the equation of the tangent line.
24. Using the point-slope form of a linear equation we substitute, simplify, and finally solve for y in order to find the equation of the tangent line as y equals negative x. You can verify this with a graphing utility, free for your computer on my website, by graphing the function and the tangent line in the same viewing window.
25. Our second example is the function square root of the quantity x minus 1 at the point 5 comma 2. We have our x and y values at the point, so all we need to find in order to write the equation of the tangent line is the slope of the function at that point. The slope of the tangent line is the derivative, so let's find the derivative at x = 5. First we find f of 5 plus h and simplify. Then we evaluate and find f of 5. Once we have our two pieces, we can take the difference in order to get the numerator of the limit for the derivative.
26. We take this numerator and put it over h as the limit tells us. But if we let h go to zero, the denominator certainly becomes zero and by quick inspection we see the numerator will also. We will have the indeterminate form 0/0 so something must be done. There is a radical involved so let's rationalize the numerator and see what happens.
27. With the rationalization of the numerator let's try the limit. The derivative of f at 5 is equal to the limit as h approaches zero of the quotient 1 over radical of the quantity 4 plus h then add 2. As h goes to zero, so we no longer need to write the limit, we get ¼ as the derivative of f at 5. Let's not forget the instructions, we need to write the equation of the tangent line, not just find the derivative. So we substitute our y, m, and x values appropriately and simplify to solve for y.
28. In our final example of this type we look at a rational function. Let's let f of x equal 1 over the quantity x plus 1 and look at the point 0 comma 1. To find the derivative at the point we must find f of 0 plus h and f of 0. Our next step is the find the difference of these expressions. When we have 1 over quantity h plus 1 then subtract 1 we need to find a common denominator. Rewriting 1 with a common denominator is straight forward, then we distribute the subtraction and simplify to get negative h over quantity h plus 1.

29. The derivative of f at zero is equal to the limit as h approaches zero of our quotient all divided by h. Complex fractions are indeed complex so we can multiply by 1 over h rather than dividing by the h in order to simplify to our next limit. As h approaches zero of negative 1 over quantity h plus 1, we get negative one over one which equals negative one. Remember, this is just our slope. The instructions ask for the equation of the tangent line through the point given. Therefore we input our y, our m, our x and then simplify to get y equals negative x plus 1 as the equation of the tangent line.
30. In this example we want to find an equation of a line that is tangent to the given function and is parallel to the given line. We're going to break this problem into three main parts. The first of which is to find the slope of the given line. We know that two parallel lines have equal slopes so if we know the slope of the given line, we'll know the slopes of the tangent line. Solving by adding y to both sides we find that m = 3. Our second main task is to find all point on the given function that have a slope of a tangent line, also known as a derivative, equal to 3.
31. In order to do this we find f of x plus h, being sure to multiply out completely x plus h multiplied by itself three times, then subtract f of x. Remember, for a polynomial at this stage we should only have terms remaining that have h's. When we form the quotient and divide by h we are left with 3 x squared plus 3xh plus h squared.
32. With this difference quotient we can find the derivative of f at x to be the limit has h approaches zero of the polynomial. But as h approaches zero, two terms get canceled out and we are left with 3 x squared. Remember, we are looking for points on the graph where the derivative (slope of the tangent line) is equal to 3, so we set the derivative equal to 3 and solve. It turns out that there are two times on the graph of f of x where the derivative is equal to 3 and those are at x values of plus and minus 1. Using the original function we can find the y values that go with each of these inputs.
33. Our third, and last, step is to finally finish the problem and write the equations of both tangent lines using appropriate x and y values with slopes of 3 in both cases.
34. As with many things in mathematics, there is more than one way to find a derivative. We have an alternate definition that looks a lot like the average rate of change formula we have seen in the past. In this definition, the derivative of f at c is given by the limit as x approaches c of quantity f of x minus f of c all divided by the difference of x minus c. This really is nothing new as the difference quotient in this case is the generalized slope formula and the limit allows us to look at nonlinear functions.
35. We're going to use the alternative form of the derivative to find the derivative of f of x equals x multiplied by the quantity x minus 1 when c = 1. Following the definition exactly we know that the derivative of f at 1 is equal to the limit as x approaches 1 of x times x minus 1 (our f of x) minus 1 times 1 minus 1 (our f of 1) all over x minus 1. Notice I didn't take this in pieces and I just put it together inside the limit. As we simplify, we find that the f(1) = 0 and we end up with the limit as x approaches 1 of x times x minus 1 all over x minus 1. But we can cancel the factors to get the limit as x approaches 1 of x which is just 1. That is, the derivative of f at 1 is equal to 1.
36. Next, let's look at a rational function f of x equals 2 over x when c equals 5. Using the definition we find that f prime of 5 equals the limit as x approaches 5 of the quantity 2 over x minus 2

fifths all over x minus 5. This looks hard. Let’s simplify. First we’ll combine the rational expressions in the numerator and then simplify the overall fraction.

37. To combine the rational expressions we need a common denominator so we multiply each appropriately. Once they have the common denominator of 5x we can combine the numerators to get the single rational expression 10 minus 2x all over 5x. This was our original numerator so now we divide it by x minus 5. But dividing by x minus 5 is the same as multiplying by 1 over x minus 5. Now we factor and cancel to simplify our complex fraction to negative 2 over 5x.
38. Back to the limit. Our derivative is equal to our original limit which is equal to the limit as x approaches 5 of the simplified version. Once we input the 5, we no longer need the limit notation, and we find the derivative of f at 5 is negative 2 over 25.
39. This is the end of the lecture on the definition of the derivative. If you would like some easy to create graphs, I recommend desmos dot com. (This is not a paid endorsement.)